The largest eigenvalues of sample covariance matrices for a spiked population: diagonal case.

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Abstract

We consider large complex random sample covariance matrices obtained from "spiked populations", that is when the true covariance matrix is diagonal with all but finitely many eigenvalues equal to one. We investigate the limiting behavior of the largest eigenvalues when the population and the sample sizes both become large. Under some conditions on moments of the sample distribution, we prove that the asymptotic fluctuations of the largest eigenvalues are the same as for a complex Gaussian sample with the same true covariance. The real setting is also considered.

1 Introduction and results

Sample covariance matrices are fundamental to multivariate statistics. Their spectral properties are e.g. important for Principal Component Analysis. In the case where the population size remains "small" while the sample size becomes sufficiently large, these spectral properties are well-understood. It is a classical probability result that the sample covariance matrix is a good approximate of the population covariance. Nowadays it is of strong interest to study the case where both the sample and population sizes become large, due to the large amount of data available. In this setting, the study of asymptotic spectral properties of sample covariance matrices has many applications. The behavior of Principal Component Analysis has first to be understood. We refer the reader to [14] and [10] for a review of other statistical applications. Other examples of applications include genetics [19], mathematical finance [1], [7], [8], [15], wireless communication [28], physics of mixture [23] and statistical learning [12].

In this paper, we investigate the limiting distribution of the largest eigenvalues

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1

of sample covariance matrices for some so-called "spiked population models". Such models have been introduced for a Gaussian sample in [14] and correspond to the case where the true covariance is a small rank perturbation of the Identity matrix. In this paper, both the impact of the largest eigenvalues of the true covariance and that of the distribution of the sample on the asymptotic behavior of the largest eigenvalues are investigated. The fluctuations of the largest eigenvalues of some non necessarily Gaussian samples are compared to those of a Gaussian sample with the same true covariance. These questions are mainly motivated by statistical applications. Indeed some statistical tests are based on the conjecture that the behavior of the largest eigenvalues of spiked sample covariance matrices is the same as for a Gaussian sample provided the sample distribution is close to a Gaussian distribution (see e.g. [19]).

1.1 Model and results

Let $X = X_N$ be a $N \times p$ complex (resp. real) random matrix such that $\{\Re e X_{ij}, \Im m X_{ij}; 1 \leq i \leq N, 1 \leq j \leq p\}$ (resp. $\{X_{ij}, 1 \leq i \leq N, 1 \leq j \leq p\}$) are real independent random variables satisfying for all $1 \leq i \leq N$ and $1 \leq j \leq p$:

(H$_1$) $\mathbb{E}X_{ij} = 0$ and $\mathbb{E}\left(\Re e X_{ij}\right)^2 = \mathbb{E}\left(\Im m X_{ij}\right)^2 = \sigma^2/2$ (resp. $\mathbb{E}X_{ij}^2 = \sigma^2$);

(H$_2$) there exists a constant $C_o > 0$ independent of N, p (and (i, j)) such that
$$\forall k > 0, \quad \mathbb{E}|X_{ij}|^{2k} \leq (C_o k)^k \, ;$$

(H$_3$) $\mathbb{E}\left((\Re e X_{ij})^{2k+1}\right) = \mathbb{E}\left((\Im m X_{ij})^{2k+1}\right) = 0$ (resp. $\mathbb{E}\left(X_{ij}^{2k+1}\right) = 0$) $\forall k \geq 0$.

To avoid technicalities, we assume throughout the paper that $p \geq N$. Here the size of the matrix X goes to infinity in such a way that if we set $\gamma_N = p/N$,

$$\exists \gamma \geq 1 \text{ such that } \gamma_N \to \gamma \text{ as } N \to \infty. \tag{1}$$

Let r be a given integer independent of N and p. Let also $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_r > 1$ be given real numbers, all of which are independent of N and p. The covariance matrix $\Sigma = \Sigma_N$ is the $N \times N$ diagonal matrix

$$\Sigma = \text{diag}(\pi_1, \pi_2, \ldots, \pi_r, 1, \ldots, 1). \tag{2}$$

The goal of this paper is to describe the large-N-limiting distribution of the largest eigenvalues of the spiked model defined by

$$V_N := \frac{1}{p}\Sigma^{1/2}XX^*\Sigma^{1/2}. \tag{3}$$

Note that the spectral properties of the associated matrix $V_N' = \frac{1}{p}X^*\Sigma X$ can be deduced from those of V_N since their non-zero eigenvalues are equal.
Throughout this paper, the white (or null) model corresponds to $\Sigma = Id$ (or $r = 0$) and is called

$$M_N := \frac{1}{p}XX^*. \tag{4}$$

2

When the entries of X are further assumed to be Gaussian random variables, we write M_N^G (resp. V_N^G) instead of M_N (resp. V_N). M_N^G is then a matrix from the so-called Laguerre unitary (resp. orthogonal) ensemble (LUE (resp. LOE)) of parameter σ, also known as the complex (resp. real) Wishart ensembles.

First, let us consider the global behavior of the spectrum. Let $\lambda_1(V_N) \geq \lambda_2(V_N) \geq \cdots \geq \lambda_N(V_N)$ be the ordered eigenvalues of V_N and let μ_N be the spectral measure defined by $\mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(V_N)}$. Setting

$$u_\pm = \sigma^2 (1 \pm \gamma^{-1/2})^2, \tag{5}$$

it is a well-known result of [16] that, for any matrix Σ given by (2) (including $\Sigma = Id$), μ_N a.s. converges as $N \to \infty$ to the Marchenko-Pastur distribution μ_{MP} whose density is $\frac{d\mu_{MP}(x)}{dx} = \frac{\gamma}{2\pi x \sigma^2} \sqrt{(u_+ - x)(x - u_-)} 1_{[u_-, u_+]}(x)$. The global behavior of the spectrum is thus not impacted by the spiked structure of Σ.

The situation is drastically different for the largest eigenvalues. Let us first recall the well-known asymptotic behavior of the largest eigenvalues of M_N. This asymptotic behavior has been identified for the complex or real Wishart ensembles M_N^G in [13] and [14] and later extended to a much wider class of white sample covariance matrices M_N in [20]. To be more precise, we need the following definitions. We denote by Ai the standard Airy function. Define the Airy kernel by $A(u, v) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u - v}$ and let A_x be the operator acting on $L^2((x, +\infty))$ with kernel $A(u, v)$. Let $F_{GU(O)E}$ be the GU(O)E Tracy-Widom distribution defined in [29], which is the limiting distribution of the largest eigenvalue of the Gaussian unitary (resp. orthogonal) ensemble (GU(O)E) as the size tends to infinity. It can in particular be shown that F_{GUE} is given by the Fredholm determinant $F_{GUE}(x) = \det(1 - A_x)$. More generally, given an integer $K \geq 1$, we denote by $F_{GU(O)E}^K$ the limiting joint distribution of the K largest eigenvalues of the GU(O)E (the precise definitions are given in [29] and [30]). Last we define

$$\rho_N = \sigma^2 \left(1 + \gamma_N^{-1/2}\right)^2 \quad \text{and} \quad \sigma_N = \gamma_N^{-1/2} \sigma^2 \left(1 + \gamma_N^{-1/2}\right)^{4/3}.$$

The next theorem has been proved in the more general case where $\gamma \in [0, \infty]$.

Theorem 1.1. *[13], [14], [20] Let $K \geq 1$ be an integer. Let M_N be a complex (or real) random matrix given by (4) and assume that X satisfies $(H_1) - (H_3)$. Then, for any $(x_1, \ldots, x_K) \in \mathbb{R}^K$, one has that*

$$\lim_{N \to \infty} \mathbb{P}\left(\frac{N^{\frac{2}{3}}}{\sigma_N}(\lambda_i(M_N) - \rho_N) \leq x_i, \, \forall i = 1, \ldots, K\right) = F_{GU(O)E}^K(x_1, x_2, \ldots, x_K).$$

Theorem 1.1 implies that a.s. $\lim_{N \to \infty} \lambda_1(M_N) = u_+$ (this result is proved in a more general setting in [31]). The situation may be quite different if Σ is

3

chosen as in (2). The first results in this direction have been obtained in [5] for the complex Gaussian spiked population model V_N^G. Therein the authors point out a phase transition phenomenon for the fluctuations of the largest eigenvalue according to the value of the largest eigenvalue(s) of the covariance matrix Σ. To state the result, further definitions are needed. For all $m \geq 1$, set

$$s^{(m)}(u) = \int_{\infty e^{\frac{5i\pi}{6}}}^{\infty e^{\frac{i\pi}{6}}} \frac{e^{\{iua + \frac{ia^3}{3}\}}}{2\pi} \frac{da}{(ia)^m}, \quad t^{(m)}(v) = \int_{\infty e^{\frac{5i\pi}{6}}}^{\infty e^{\frac{i\pi}{6}}} \frac{e^{\{iva + \frac{ia^3}{3}\}}}{2\pi} (ia)^{m-1} da,$$

which are integrals or derivatives of the standard Airy function (cf. [5]). For any $k \geq 1$, define the distribution function F_k (see [4]) by

$$F_k(x) = \det(1 - A_x) \det\left(\delta_{m,n} - < \frac{1}{1 - A_x} s^{(m)}, t^{(n)} >\right)_{1 \leq m,n \leq k}, \qquad (6)$$

for any real x, where $< ., . >$ denotes the (real) inner product of functions in $L^2((x, \infty))$. Define also

$$w_c := 1 + \frac{1}{\sqrt{\gamma}}, \qquad (7)$$

$$\tau(\pi_1) = \sigma^2 \pi_1 \left(1 + \frac{\gamma^{-1}}{\pi_1 - 1}\right), \quad \sigma(\pi_1) = \sigma^2 \pi_1 \sqrt{1 - \gamma^{-1}/(\pi_1 - 1)^2}. \quad (8)$$

Note that if $\pi_1 > w_c$ (resp. $\pi_1 = w_c$, resp. $\pi_1 < w_c$), then $\tau(\pi_1) > u_+$ (resp. $\tau(\pi_1) = u_+$, resp. $\tau(\pi_1) < u_+$).

We here give only the asymptotic behavior of the largest eigenvalue of the complex non-white Wishart ensemble (see Remark 1.1 for some extentions).

Theorem 1.2. *[5] Consider the sequence of complex Wishart matrices (V_N^G) when Σ is given by (2). Let $1 \leq k \leq r$ be an integer. For any real x, one has that*

(i) *If $\pi_1 = \ldots = \pi_k > w_c$ and $\pi_{k+1} < \pi_1$ then*

$$\lim_{N,p \to \infty} \mathbb{P}\left(\frac{\sqrt{N}}{\sigma(\pi_1)} \left(\lambda_1(V_N^G) - \tau(\pi_1)\right) \leq x\right) = G_k(x),$$

where G_k is the distribution of the largest eigenvalue of the un-normalized GUE random matrix $H = (H_{ij})_{i,j=1}^k$ with i.i.d. complex standard Gaussian entries above the diagonal.

(ii) *If $\pi_1 = \ldots = \pi_k = w_c$ and $\pi_{k+1} < w_c$ then*

$$\lim_{N,p \to \infty} \mathbb{P}\left(\frac{N^{2/3}}{\sigma_N} \left(\lambda_1(V_N^G) - \rho_N\right) \leq x\right) = F_k(x).$$

(iii) *If $\pi_1 < w_c$ then* $\lim_{N,p \to \infty} \mathbb{P}\left(\frac{N^{2/3}}{\sigma_N} \left(\lambda_1(V_N^G) - \rho_N\right) \leq x\right) = F_{\text{GUE}}(x).$

4

Remark 1.1. The joint distribution of the K largest eigenvalues ($K \leq k$ in (i)) can easily be deduced by a straightforward extension of the arguments of [5] in cases (i) and (iii). In case (i), the joint distribution of the K (correctly rescaled) largest eigenvalues $\lambda_i(V_N^G), 1 \leq i \leq K$, converges to the law of the K largest eigenvalues of H (see also [21]). In case (iii), the full conclusion of Theorem 1.1 holds true.

Some extensions of Theorem 1.2 have been obtained. First, in [17], the counterpart of Theorem 1.2 has been established for singular Wishart matrices. Therein, the case where $p < N$ and $\lim_{N \to \infty} \gamma_N = \gamma \in [0,1]$ is investigated. The same phase transition phenomenon is established. In [18], real Wishart ensembles are considered. Unlike complex (singular or not) Wishart ensembles, the joint eigenvalue density is not known in the real setting. Using perturbation theory, it is proved that when $\pi_1 > w_c$ is simple, the largest eigenvalue of real non-white Wishart matrices V_N^G exhibits Gaussian fluctuations (with a different variance). Some more recent extensions have also been obtained in [3] and are recalled below.

In view of Theorem 1.1, it is natural to investigate the question whether Theorem 1.2 (and its real analogue) would actually hold true for non Gaussian samples. Our main results, exposed below in Theorem 1.5 and Theorem 1.6, answer this universality question. Before that, a partial answer has been given at the level of a.s. convergence by [6]. We partially state here their result.

Theorem 1.3. *[6] Let V_N be a complex or real sample covariance matrix defined by (3) with Σ given by (2). Assume that the entries of X are i.i.d. with $\mathbb{E} X_{11} = 0$, $\mathbb{E}|X_{11}|^2 = \sigma^2$ and $\mathbb{E}|X_{11}|^4 < \infty$. Let $1 \leq k \leq r$ be an integer. Then,*

   (i) If $\pi_1 = \ldots = \pi_k > w_c$ and $\pi_{k+1} < \pi_1$ then $\lambda_1(V_N), \ldots, \lambda_k(V_N)$ a.s. converge to $\tau(\pi_1)$.

   (ii) If $\pi_1 \leq w_c$ then $\lambda_1(V_N)$ a.s. converges to u_+.

Regarding the fluctuations of the largest eigenvalues now, [3] determines their asymptotic distribution in the case where these eigenvalues are well separated from the bulk. They consider both the complex and real models in the case where the entries of X are i.i.d. with finite fourth moment. They prove that (i) in Theorem 1.2 holds true for a wide class of non Gaussian samples X only if the rescaling factor $\sigma(\pi_1)$ given in (8) is modified to include the "excess kurtosis" of the entries of X. The excess kurtosis is given by $\frac{\mathbb{E}|X_{11}|^4}{\sigma^4} - 2$ (resp. $\frac{\mathbb{E}|X_{11}|^4}{\sigma^4} - 3$) in the complex (resp. real) case and is zero for Gaussian distributions. One may also indicate that their result is stated in a more general setting than that considered here: in particular, the true covariance does not need to be diagonal. To ease the exposition, we give their result with the added condition (9) which requires the fourth moment of X_{11} to be as in the Gaussian case.

Theorem 1.4. *[3] Assume that the assumptions of Theorem 1.3 are satisfied with*

$$\mathbb{E}(X_{11}^4) = (1 + \beta')\sigma^4 \tag{9}$$

where $\beta' = 1$ (resp. $\beta' = 2$) in the complex (resp. real) case. Define

$$\xi_i(V_N) := \frac{\sqrt{N}}{\sigma(\pi_1)} \left(\lambda_i(V_N^G) - \tau(\pi_1) \right).$$

If $\pi_1 = \ldots = \pi_k > w_c$ and $\pi_{k+1} < w_c$ then the N-limiting distribution of $(\xi_1(V_N), \ldots, \xi_k(V_N))$ is the joint distribution of the k eigenvalues, ordered in the decreasing order, of the GUE (resp. GOE) $H = (H_{ij})_{i,j=1}^k$ with i.i.d. complex (resp. real) standard Gaussian entries above the diagonal.

The approach of [3], following ideas of [18], is mainly based on the fact (contained in Theorem 1.3) that the largest eigenvalues split from the bulk. The proof relies on some perturbation theory ideas which allow to see the rescaled largest eigenvalues $\xi_i(V_N)$ as the eigenvalues of a $k \times k$ random matrix defined in terms of the resolvent of an underlying white matrix. The conclusion then essentially follows from a CLT on random sesquilinear forms, explaining the assumption on the first four moments of the entries X_{ij}.

The question of the universality of the two other regimes in Theorem 1.2 remains open. This is the gap we here fill in and is the main result of this note.

Theorem 1.5. *Consider the sequence of complex sample covariance matrices (V_N) defined by (3) where the entries of X satisfy $(H_1) - (H_3)$ and Σ is given by (2). Let $1 \le k \le r$ be an integer.*
When $\pi_1 \ge w_c$, assuming furthermore that

$$(H_4) \quad \mathbb{E}(\Re e X_{ij})^4 = \mathbb{E}(\Im m X_{ij})^4 = 3\sigma^4/4, \quad \forall 1 \le i \le N, \forall 1 \le j \le p,$$

the conclusions of Theorem 1.2 hold true for V_N.
When $\pi_1 < w_c$, the conclusion of Theorem 1.1 is true for V_N.

In the real setting, we prove the universality in the two non-critical regimes.

Theorem 1.6. *Consider the sequence of real sample covariance matrices (V_N) defined by (3) where the entries of X satisfy $(H_1) - (H_3)$ and Σ is given by (2). Let $1 \le k \le r$ be an integer.*
When $\pi_1 > w_c$, assuming furthermore that

$$(H_4') \quad \mathbb{E}(X_{ij}^4) = 3\sigma^4, \quad \forall 1 \le i \le N, \forall 1 \le j \le p,$$

the conclusion of Theorem 1.4 hold true for V_N.
When $\pi_1 < w_c$, the conclusion of Theorem 1.1 is true for V_N.

As we will explain, the proof of these theorems is based on a combinatorial method combined with some results on the corresponding Gaussian model. In fact, our following combinatorial arguments also cover the real setting of Theorem 1.6 in the critical case where $\pi_1 = w_c$. Thus, our analysis reduces the universality problem in this case (under the assumption (H_4')) to the knowledge of the asymptotic fluctuations of the largest eigenvalues of the associated real Gaussian model. Unfortunately the latter result is not known so far.

1.2 Core of the proof

We here give the main ideas of the proof of Theorems 1.5 and 1.6. We first mainly concentrate on the complex setting. At the end of this section, we discuss the main modifications needed to consider the real case.

The proof follows essentially the strategy introduced in [20] (see also [26] and [27]) and we refer to this paper for most of the detail. We also refer the reader to [11] where the authors investigate Deformed Wigner matrices which are classical Wigner matrices decentered by a particular deterministic matrix. The Deformed Wigner model can be seen as the additive analogue of the present model. In particular, it exhibits a similar phase transition phenomenon regarding the asymptotic behavior of the largest eigenvalues. [11] establishes the universality of the fluctuations of the largest eigenvalues for non-necessarily Gaussian Deformed Wigner matrices. The approach developed here is close to that of [11] and is mainly based on combinatorial arguments.

Basically, and for each of the three regimes depending on the value of π_1, we compute the leading term in the asymptotic expansion of expectations (and also higher moments) of traces of high powers of V_N that is $\mathbb{E}(\mathrm{Tr}V_N^{s_N})$ where Tr denotes the classical (un-normalized) trace. We consider specific exponents s_N which depend on the scaling of the fluctuations of the largest eigenvalue(s) when the size N goes to ∞. The core of the proof is to show the universality of moments (of any fixed order) of traces of powers of V_N in these scales. Let us explain this more precisely in the particular case of the expectation.

In the case where $\pi_1 > w_c$, it is expected that the largest eigenvalue(s) exhibits fluctuations in the scale $N^{-1/2}$ around $\tau(\pi_1)$. We thus consider an arbitrary sequence of integers (s_N) such that $\lim_N s_N/N^{1/2} = c$ for some constant $c > 0$. We first show that $\mathbb{E}(\mathrm{Tr}(V_N/\tau(\pi_1))^{s_N})$ is bounded. Then, we prove that the leading term in the asymptotic expansion of $\mathbb{E}(\mathrm{Tr}V_N^{s_N})$ depends on π_1, on the variance σ^2 and on the fourth moment of the X_{ij}'s only. Assume now that (H_4) is satisfied i.e. the fourth moment of the X_{ij}'s is taken to be that of the Gaussian distribution with variance σ^2. Then, up to a negligible error, the expectation $\mathbb{E}(\mathrm{Tr}V_N^{s_N})$ does not depend asymptotically on the particular law of the entries and one has that

$$\mathbb{E}(\mathrm{Tr}V_N^{s_N}) = \mathbb{E}[\mathrm{Tr}(V_N^G)^{s_N}](1 + o(1)). \tag{10}$$

In the critical case where $\pi_1 = w_c$, the largest eigenvalue fluctuates now in the scale $N^{-2/3}$ around the right-edge u_+ of the Marchenko-Pastur support. The powers s_N to be considered are such that $\lim_N s_N/N^{2/3} = c$ for some constant $c > 0$ and we first show that $\mathbb{E}(\mathrm{Tr}(V_N/u_+)^{s_N})$ is bounded. We then prove that the leading term in the asymptotic expansion of $\mathbb{E}(\mathrm{Tr}V_N^{s_N})$ depends on σ^2 and on the fourth moment of the entries of X only. So assuming again that (H_4) holds true, we show that the expectation behaves in the large N-limit as in the Gaussian case and that (10) still holds true.

In the sub-critical case $\pi_1 < w_c$, we still consider powers s_N in the order of $N^{2/3}$. Here, the fluctuations of the largest eigenvalues are expected not to

depend on π_1 and to be exactly as in the white case where $\Sigma = Id$. We get this by showing that $\mathbb{E}(\mathrm{Tr}V_N^{s_N}) = \mathbb{E}[\mathrm{Tr}M_N^{s_N}](1 + o(1))$. Thanks to the investigations of [20] on the universality of the fluctuations of the largest eigenvalues of the white matrices M_N, we can deduce that

$$\mathbb{E}(\mathrm{Tr}V_N^{s_N}) = \mathbb{E}[\mathrm{Tr}(M_N^G)^{s_N}](1 + o(1)). \tag{11}$$

Actually, we prove universality of all the moments (of fixed order) of traces of high powers of V_N as in (10) and (11). Using the machinery developed in [26] (Sections 2 and 5) and [27] (Section 2), we can then deduce that the limiting distribution of the largest eigenvalue(s) for spiked population matrices V_N satisfying (H$_1$)–(H$_3$) as well as (H$_4$) if $\pi_1 \geq w_c$, is the same as for complex non-white Wishart matrices V_N^G. When $\pi_1 < w_c$, we more generally get the universality of the limiting joint distribution of any fixed number of largest eigenvalues. Let us roughly give the main ideas. On the one hand, the Laplace transform of the joint distribution of a finite number of the (correctly rescaled) largest eigenvalues of V_N can be conveniently expressed in terms of joint moments of traces of the matrix V_N taken at suitable powers s_N (those of the previous discussion). On the other hand, the asymptotic distribution of the rescaled largest eigenvalues (and also the corresponding Laplace transform) is well-known in the complex Gaussian setting. One can then deduce from universality of moments of traces that the asymptotic joint distribution of the largest eigenvalues for any model V_N considered here is the same as for the corresponding Gaussian case. The detail of the derivation of such a result from formulas (10) and (11), including the required asymptotics of correlation functions for the complex non-white Wishart matrix V_N^G, can be found in [11], [5], [26] and [27].

In the real setting, our combinatorial reasoning also yields the universality of moments of traces of high powers of V_N. This could be used in principle to prove universality of the fluctuations of the largest eigenvalues, provided the full counterpart of Theorem 1.2 in the real Gaussian case was fully established. This is true in the case where the largest eigenvalue of the true covariance is simple and satisfies $\pi_1 > w_c$ (cf. [18]). The non Gaussian case is actually also covered by Theorem 1.4. We can come to the same conclusion (assuming (H$'_4$)) using our approach. In the sub-critical case where $\pi_1 < w_c$, we are also able to conclude thanks to (11) (and its analogue for higher moments) which proves that the fluctuations of eigenvalues of V_N are similar to those, well-known, of the real Wishart matrix M_N^G. In the critical case $\pi_1 = w_c$, we cannot conclude.

Our paper is organized as follows. In Section 2, we introduce the major combinatorial tools needed to compute moments of traces of high powers of V_N. We first recall the specific terminology and the main arguments (Section 2.2) developed by [20] for the investigations of the white case. We then present (Section 2.3) the main ideas of the strategy we will use to deal with the non-white case when $r = 1$. In Section 3, we establish the universality of the asymptotic expectation of traces of high powers of V_N. We next consider higher moments

in Section 4. In Section 5, we discuss the main modifications needed to deal with the case where $r > 1$.

2   Combinatorial tools

In this section, we define the major combinatorial tools needed to investigate the asymptotics of moments of traces of large powers of V_N. We here extend some of the tools used in [20] where the white case ($\Sigma = Id$) is investigated. The reader is referred to Sections 2 and 3 of the above cited article for a detailed explanation of the following combinatorial approach. We here choose to explain our strategy in the case where $r = 1$ and thus consider the covariance matrix

$$\Sigma = \mathrm{diag}(\pi_1, 1, 1, \ldots, 1).$$

Modifications to handle more complex cases ($r > 1$) are indicated in Section 5. Thoughout the paper, we denote by $C, C', C_i, i = 1, 2, \ldots$ some positive constants independent of N and whose value may vary from line to line.

2.1   Paths and 1-edges

Let (s_N) be a sequence of integers that may grow to infinity. Developing the trace, one obtains that

$$\mathbb{E}\big[\mathrm{Tr}\, V_N^{s_N}\big] \tag{12}$$

$$= \frac{1}{p^{s_N}} \sum_{i_0, i_1, \ldots, i_{s_N-1}} \pi_1^{\sum_{q=0}^{s_N-1} \delta_{1 i_q}} \mathbb{E}\left(\prod_{q=0}^{s_N-1} (XX^*)_{i_q i_{q+1}}\right)$$

$$= \frac{1}{p^{s_N}} \sum_{i_0, i_1, \ldots, i_{s_N-1}} \pi_1^{\#\{q:\, i_q=1\}} \sum_{j_1, \ldots, j_{s_N}} \mathbb{E}\left(\prod_{q=0}^{s_N-1} X_{i_q j_{q+1}} \overline{X_{i_{q+1} j_{q+1}}}\right) \tag{13}$$

$$\text{where } i_q \in \{1, 2, \ldots, N\} \quad \text{and} \quad j_q \in \{1, 2, \ldots, N, \ldots, p\} \tag{14}$$

and where we use the convention that $i_{s_N} = i_0$.

To each term $\prod_{q=0}^{s_N-1} X_{i_q j_{q+1}} \overline{X_{i_{q+1} j_{q+1}}}$ in (13), we associate three combinatorial objects needed in the following. First, we define the "edge path" $\mathcal{P}$ formed with oriented edges (read from bottom to top) by

$$\mathcal{P} = \begin{pmatrix} j_1 \\ i_0 \end{pmatrix} \begin{pmatrix} j_1 \\ i_1 \end{pmatrix} \begin{pmatrix} j_2 \\ i_1 \end{pmatrix} \cdots \begin{pmatrix} j_{s_N} \\ i_{s_N-1} \end{pmatrix} \begin{pmatrix} j_{s_N} \\ i_0 \end{pmatrix}. \tag{15}$$

Due to the symmetry assumption (H$_3$), only paths for which any oriented edge occurs an even number of times give a non zero contribution to the expectation. From now on, we only consider such even paths.

To an arbitrary even edge path $\mathcal{P}$, we associate a so-called Dyck path (Dyck paths have a long history in Random Matrix Theory, see [32], [2] for instance), that is a trajectory $x = \{x(t), t \in [0, 2s_N]\}$ on the positive half-lattice such that:

$$x(0) = x(2s_N) = 0; \quad \forall t \in [0, 2s_N], \, x(t) \geq 0 \ \text{ and } \ x(t) - x(t-1) = \pm 1.$$

To define the Dyck path x associated to $\mathcal{P}$, we read the oriented edges of $\mathcal{P}$ in the order of appearance and draw an up (resp. down) step $(1, +1)$ (resp. $(1, -1)$) if the current edge is read for an odd (resp. even) number of times. Last, we also associate to $\mathcal{P}$ a "usual" path denoted by P : we mark on the underlying Dyck path x the successive vertices met in $\mathcal{P}$ and then set $P := i_0 \, j_1 \, i_1 \, j_2 \ldots i_0$.

The strategy in the rest of the paper can roughly be summarized as follows. Given a trajectory x, we shall estimate the number of edge paths that can be associated to x and then we shall estimate their contribution to the expectation (12). On the one hand, due to the constraint (14) on the choice of the vertices, we shall refine the enumeration of Dyck paths according to the number of odd up steps. The way to handle such a specificity has been developed in detail in [20]. We recall some points of the analysis made by [20] in the next subsection. On the other hand, when estimating the contribution of such edge paths, we also have to take into account the occurrences of the vertex 1 on the bottom line since each occurrence yields an additional weight π_1 (recall (13)). To this aim, we introduce the notion of 1-edge.

Definition 2.1. *A* 1-*edge is an oriented edge with* 1 *on the bottom line i.e. an edge* $\begin{pmatrix} h \\ 1 \end{pmatrix}$ *where* $h \in \{1, \ldots, p\}$.

Remark 2.1. An edge $\begin{pmatrix} 1 \\ h \end{pmatrix}$ is not a 1-edge in our denomination if $h \in \{2, \ldots, N\}$.

Thus, we shall be able to refine the analysis made in [20] to estimate the contribution of edge paths with 1-edges.

The rest of this section is organized as follows. In Subsection 2.2, we recall the main definitions and results of [20] that we will use throughout this paper. Note that the investigations of [20] readily gives the contribution to the expectation (12) of edge paths without 1-edges (see Proposition 2.2). In Subsection 2.3, we explain the main ideas of the strategy we will use to deal with paths with 1-edges and compute the corresponding contribution to the expectation (12).

2.2 The white case $\Sigma = Id$ and paths with no 1-edges

The aim of this section is to recall the main definitions and results derived from [20] that we will use throughout this paper. These results also allow us to

estimate at the end of this subsection the contribution to (12) of paths without 1-edges. We assume some familiarity of the reader with the combinatorial machinery developed in refs. [24], [25], [26] and [20].

Notational Remark. Our notations differ from those used in [20] : the (white) model considered by [20] corresponds here to $\frac{\sigma^{-2}}{N} X X^*$ that is $\sigma^{-2} \gamma_N M_N$.

To handle the case where $\Sigma = Id$ (that is the computations of $\mathbb{E}\mathrm{Tr} M_N^{s_N}$), [20] first enumerates the associated Dyck paths according to the number of odd up steps. Thus, we let $\chi_{s_N,k}$ be the set of Dyck paths of length $2s_N$ with k odd up steps and $\chi_{s_N} = \cup_{i=1}^{s_N} \chi_{s_N,k}$ be the set of Dyck paths of length $2s_N$.

Definition 2.2. *[9] Let $\mathbf{N}(s_N, k)$ be the kth Narayana number defined by*

$$\mathbf{N}(s_N, k) = \frac{1}{s_N} \binom{s_N}{k} \binom{s_N}{k-1}. \tag{16}$$

Then $\mathbf{N}(s_N, k) = \sharp \mathcal{X}_{s_N,k}$.

The reader is referred to [20] and references therein for further detail about Narayana numbers.

Given a Dyck path $x \in \mathcal{X}_{s_N,k}$ we shall estimate the number of edge paths associated to it. First, one needs to assign a vertex from $\{1, \ldots, N\}$ (resp. $\{1, \ldots, p\}$) to each even (resp. odd) moment of time along the Dyck path x. For this, we need the following definition.

Definition 2.3. *An instant $t \in [1, 2s_N]$ is said to be marked (in x or in P) if it corresponds to the right endpoint of an up edge.*

Roughly speaking, marked instants correspond to the moments of time (apart from $t = 0$) where one can discover in P a vertex never encountered before. Thus, in order to estimate the number of paths that can be associated to a given trajectory x, we first choose the vertices occurring at the marked instants, which we call marked vertices, and at the origin of the path (which is non marked by definition).

In order to choose the vertices occurring at the marked instants, we refine our classification separating the cases where they are on the bottom or top line in $\mathcal{P}$, that is the cases where they are marked at even or odd instants in P, as follows. For any integer $0 \leq i \leq s_N$, we define two classes of marked vertices: $\mathcal{N}_i = \{$ vertices occuring i times at an even marked instant $\}$ and $\mathcal{T}_i = \{$ vertices occuring i times at an odd marked instant $\}$ and we set $n_i = \sharp \mathcal{N}_i$ and $p_i = \sharp \mathcal{T}_i$. Then (cf. (14)), vertices encountered along a path P at the odd (resp. even) instants split into the disjoint classes $\mathcal{T}_i$ (resp. $\mathcal{N}_i$). Observe that $n_i = 0$, $\forall i > s_N - k$ with $\sum_i n_i = N$ and $\sum_i i n_i = s_N - k$. Similarly, one has that $p_i = 0$, $\forall i > k$ with $\sum_i p_i = p$ and $\sum_i i p_i = k$. A path P can then be characterized by its associated trajectory $x \in \chi_{s_N,k}$ for some integer $1 \leq k \leq s_N$, and its type:

$$(n_0, n_1, \ldots, n_{s_N-k})(p_1, \ldots, p_k) := (\tilde{n}, \tilde{p}).$$

Definition 2.4. *A vertex $v \in \mathcal{N}_i$ (resp. $v \in \mathcal{T}_i$) is said to be of type i on the bottom (resp. top) line.*
Any vertex $v \in \cup_{i \geq 2} \mathcal{N}_i$ (resp. $v \in \cup_{i \geq 2} \mathcal{T}_i$) is said to be a vertex of self-intersection on the bottom (resp. top) line.

In the following, $M_1 = \sum_{i \geq 2}(i-1)n_i$ (resp. $M_2 = \sum_{i \geq 2}(i-1)p_i$) denotes the number of vertices of self-intersection on the bottom (resp. top) line.

It is an easy fact that, given the type $(\tilde{n}, \tilde{p})$, the number of ways to assign vertices at the marked instants and choose the origin is at most

$$N \frac{N!}{\prod_{i=0}^{s_N - k} n_i!} \frac{p!}{\prod_{i=0}^{k} p_i!} \frac{(s_N - k)!}{\prod_{i \geq 2}(i!)^{n_i}} \frac{k!}{\prod_{i \geq 2}(i!)^{p_i}}. \tag{17}$$

Once marked vertices are chosen, there remains to count the number of ways to fill in the blanks of the path i.e. assign vertices at the unmarked instants and evaluate the corresponding expectation of each "filled path". Due to self-intersections, there may be many ways to fill in the blanks of the path as well as edges seen many times. Actually the bound (17) would be enough for the following as long as $s_N = o(\sqrt{N})$. It needs to be refined for higher scales s_N. In particular and as explained in the beginning of Section 3.2 in [20], one must pay attention to vertices of type 2. In (17), we used the rough estimate that if $t \in \{1, \ldots, k\}$ (resp. $t \in \{1, \ldots, s_N - k\}$) is an odd (resp. even) marked instant where the second occurrence of a vertex of type 2 is repeated, there are at most $t - 1$ possible choices for the vertex to be repeated. This rough estimate needs to be refined when considering vertices v of type 2 belonging to edges seen more than twice and vertices v of type 2 for which there are multiple ways to close an edge with v as its left endpoint at an unmarked instant.
Let us first consider the latter class of vertices of type 2. Note that for such a vertex v, there are at most three possible ways to close an edge with v as left endpoint at an unmarked instant. To investigate this class, we need a few definitions from [20].

Definition 2.5. *A vertex v of type 2 is said to be non-MP-closed if it is an odd (resp. even) marked instant and if there is an ambiguity for closing an edge at an unmarked instant starting from this vertex on the top (resp. bottom) line.*

Let t be a given marked instant. Assume that the marked vertices before t have been chosen and that, at the instant t, there is a non-MP-closed vertex. Then, by definition of x and of non-MP-closed vertices, there are at most $x(t)$ possible choices for this vertex. This can be checked as in [25], p. 122.
Let us turn to vertices of type 2 which belong to an edge that is read four times or more in the path. To consider such vertices, we need to introduce other characteristics of the path. Let $\nu_N := \nu_N(P)$ be the maximal number of vertices that can be visited at marked instants from a given vertex of the path P. Let also $T_N := T_N(P)$ be the maximal type of a vertex in P. Then, if at the instant t, one reads for the second time an oriented up edge e, there are at most

$2(\nu_N + T_N)$ choices for the vertex occurring at the instant t. Indeed, one shall look among the oriented edges already encountered in the path and for which one endpoint is the vertex occurring at time $t-1$ (see the Appendix in [25] and Section 5.1.2 in [11] e.g.).

Furthermore, the machinery developed by [24], [25], [26] and [20] shows that once the marked vertices are assigned along a trajectory x and once the origin is chosen, the number of filled paths, weighted by their expectation, which are associated to x (and of type $(\tilde{n}, \tilde{p})$) is bounded by

$$\frac{2\sigma^{2s_N}}{p^{s_N}} \prod_{l=3}^{s_N-k} (C_1 l)^{ln_l} \prod_{m=3}^{k} (C_1 m)^{mp_m} 3^{r_1+r_2} C_2^{q_1} C_3^{q_2}, \tag{18}$$

where the extra factor 2 comes from the negligible case where the origin i_0 is marked; the C_i's are positive constants independent of k, p, N and s_N; $r_i, i = 1, 2$ (resp. $q_i, i = 1, 2$) count the number of vertices of type 2 on the bottom/top line which are non-MP-closed (resp. belong to an edge seen more than twice).

Combining the above with some ideas previously developed in the above cited papers, it is shown in Section 3 in [20] that the contribution to $\mathbb{E}\mathrm{Tr}M_N^{s_N}$ from paths with k odd marked instants and of type $(\tilde{n}, \tilde{p})$ is bounded by

$$C\mathbf{N}(s_N, k)N\gamma_N^{k-s_N}\sigma^{2s_N} \sum_{r_1, r_2, q_1, q_2, n_i, p_i, i \geq 2} e^{\{-\frac{(s_N-k-M_1)^2}{2N} - \frac{(k-M_2)^2}{2p}\}} \mathbb{E}_k\Bigg[$$

$$\frac{\left(3(s_N - k)\max x(t)\right)^{r_1}}{r_1!} \frac{\left(C_1(s_N - k)(\nu_N + T_N)\right)^{q_1}}{q_1!} \prod_{i \geq 3} \frac{\left(\frac{C^i(s_N-k)^i}{N^{i-1}}\right)^{n_i}}{n_i!}$$

$$\frac{\left(\frac{(s_N-k)^2}{2}\right)^{n_2-r_1-q_1}}{(n_2 - r_1 - q_1)!} \frac{\left(3k\max x(t)\right)^{r_2}}{r_2!}$$

$$\frac{\left(\frac{k^2}{2}\right)^{p_2-r_2-q_2}}{(p_2 - r_2 - q_2)!} \frac{\left(C_2 k(\nu_N + T_N)\right)^{q_2}}{q_2!} \prod_{i \geq 3} \frac{1}{p_i!}\left(\frac{C^i k^i}{p^{i-1}}\right)^{p_i}\Bigg], \tag{19}$$

where
- C, C_1 and C_2 are positive constants independent of k, p, N and s_N;
- $\max x(t)$ is the maximal level reached by a trajectory x;
- $\mathbb{E}_k$ is the expectation with respect to the uniform distribution on $\mathcal{X}_{s_N,k}$.
Moreover, [20] (Section 3) establishes important estimates on the previous quantities. Proposition 3.1 in [20] proves that typical paths (that is paths which contribute in a non negligible way to $\mathbb{E}\mathrm{Tr}M_N^{s_N}$) satisfy the following constraints:

a) the number k of odd marked instants lies in the interval $[\alpha's_N, \alpha s_N]$ for any α', α such that $0 < \alpha' < \frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} < \alpha < 1$;

b) $\nu_N + T_N << \sqrt{s_N}$;

13

c) there exists a constant $c > 0$, independent of k, p, N and s_N, such that $M_1 + M_2 \leq c\sqrt{s_N}$.

Besides, [20] also proves that $\max x(t) \sim \sqrt{s_N}$ in typical paths. More precisely, it is shown in Lemma 3.1 in [20] that

$$\forall a > 0, \exists C(a) < \infty, \max_{\alpha' s_N \leq k \leq \alpha s_N} \mathbb{E}_k \left[\exp\left\{a \max x(t)/\sqrt{s_N}\right\}\right] \leq C(a). \quad (20)$$

All these results combined with (19) lead to one of the main results of Section 3 in [20].

Proposition 2.1. *If $\Sigma = Id$ and $s_N = O(N^{2/3})$, the paths with k odd marked instants contributing to $\mathbb{E}\mathrm{Tr}M_N^{s_N}$ in a non-negligible way have edges read only twice, a non marked origin and no vertex of type strictly greater than 3. Furthermore, there exists a constant $C > 0$ such that their contribution is at most*

$$\mathbf{N}(s_N, k)\, N\, \gamma_N^{k-s_N}\, \sigma^{2s_N}\, \mathbb{E}_k\left[\exp\left(6\frac{s_N \max_t x(t)}{N}\right)\right] \exp\left(C\frac{s_N^3}{N^2}\right). \quad (21)$$

From these computations, one readily deduces that the contribution to the expectation (12) from paths without 1-edges is characterized as follows.

Proposition 2.2. *The typical contribution to the expectation (12) from paths with no 1-edges is at most of the order of*

$$\sum_{\alpha' s_N \leq k \leq \alpha s_N} \mathbf{N}(s_N, k)\, N\, \frac{\gamma_N^k}{\gamma_N^{s_N}}\, \sigma^{2s_N}\, \mathbb{E}_k\left[\exp\left(6\frac{s_N \max_t x(t)}{N}\right)\right] e^{C\frac{s_N^3}{N^2}} \quad \leq \quad C' u_+^{s_N},$$

where C and C' are positive constant independent of N. Typical paths amongst those without 1-edges have edges read only twice, a non marked origin and vertices of type at most 3.

Proof of Proposition 2.2: Here the vertices i_j's must be chosen from the set $\{2, \ldots, N\}$ instead of $\{1, \ldots, N\}$, since the vertex 1 is assumed not to occur on the bottom line. Formula (19) must be simply multiplied by a factor $(1 - N^{-1})^{s_N - k + 1}$. This has no impact on the final result for N large enough. Proposition 2.2 follows by summation on the typical k's and the fact that $\sum_{k=1}^{s_N} \mathbf{N}(s_N, k)\, N\, \gamma_N^{k-s_N} = O\left((u_+/\sigma^2)^{s_N}\right)$ (see Remark 2.4 in [20]). $\square$

Notational Remark. From now on, we simplify the notations and use P to denote a usual path as well as its associated edge path $\mathcal{P}$.

We shall now be able to refine the counting procedure of [20] to estimate the contribution to the expectation (12) from edge paths (15) with 1-edges. The problem of evaluating directly the number of 1-edges occurring in a path turns out to be difficult. Thus, our strategy will be indirect. Instead of directly evaluating the contribution of paths P as well as the number of its 1-edges, we first evaluate the contribution of paths with a prescribed number of 1-edges. This is the aim of the following subsection.

2.3 Counting the number of 1-edges

In this subsection, we define a procedure called *gluing procedure* which allows to enumerate the paths P according to the number of their 1-edges. One can first notice that the contribution of a path P with 1-edges to the expectation (12) is a weighted term related to its contribution to the expectation $\mathbb{E}[\mathrm{Tr}M_N^{s_N}]$, where $M_N = \frac{1}{p}XX^*$ is the associated white matrix. One simply assigns a weight π_1 to each occurrence of the vertex 1 on the bottom line of P. Consider for a while a path P of length $2s_N$ having $s \geq 1$ pairs of 1-edges and with 1 for origin. The basic idea is that the vertex 1 necessarily occurs on the bottom line of P at the instants where the trajectory x of P hits the level 0. If one furthermore assumes that x hits exactly s times the level 0, then all the occurrences of 1 on the bottom line of P correspond to the returns to 0 of x. Thus the enumeration of 1-edges transfers to statistics on the number of returns to 0 for Dyck paths. This observation is the basic idea of the gluing procedure.

Throughout the paper, we denote by $2s$ the number of 1-edges of a path P. Starting from a general path P of length $2s_N$, the gluing procedure associates a new path P' with origin 1 as we now explain.

2.3.1 Subpaths starting and ending with a 1-edge

We denote by $\binom{h_i}{1} \binom{g_i}{1}$, $i = 1, \ldots, s$ the pairs (not necessarily distinct) of successive 1-edges occuring in P. One can then write P as

$$P = \underbrace{\binom{j_1}{i_0} \binom{j_1}{i_1} \cdots \binom{h_1}{1}} \underbrace{\binom{g_1}{1} \cdots \binom{h_2}{1}} \underbrace{\binom{g_2}{1}} \cdots \cdots \underbrace{\binom{h_s}{1}}$$

$$\underbrace{\binom{g_s}{1} \cdots \binom{j_{s_N}}{i_{s_N-1}} \binom{j_{s_N}}{i_0}}.$$

Using these 1-edges, P splits into s subpaths $P_i, i = 1, \ldots, s$, defined as follows. For $i = 2, \ldots, s$, we call P_i the subpath starting at $\binom{g_{i-1}}{1}$ and ending at $\binom{h_i}{1}$. Let then P_1 be the subpath $\binom{g_s}{1} \cdots \binom{j_{s_N}}{i_{s_N-1}} \binom{j_{s_N}}{i_0} \cdot \binom{j_1}{i_0} \binom{j_1}{i_1} \cdots \binom{h_1}{1}$.

Remark 2.2. In the particular case where $i_0 = 1$, the edge path reads as

$$P = \underbrace{\binom{j_1}{1} \binom{j_1}{i_1} \cdots \binom{h_1}{1}} \underbrace{\binom{g_1}{1} \cdots \binom{h_2}{1}} \underbrace{\binom{g_2}{1}} \cdots \cdots \underbrace{\binom{h_s}{1}}.$$

The sole difference here is that $g_s = j_1$ and P_1 is the subpath beginning P.

Let $t_1, \ldots, t_s$ denote the instants at which the successive s pairs of 1-edges occur in P. The length l_i of each subpath is determined by the instants t_i since $l_i = t_{i+1} - t_i$ (note that these lengths are necessarily even). One can also

note that the 1-edges occurring in the path are necessarily even. Thus, for any $1 \leq i \leq s$, there exists j such that $h_i = h_j$ or $h_i = g_j$. This observation is crucial for the sequel. As we will explain in the following subsection, this allows us to re-order the paths P_i and erase some of the 1-edges.

2.3.2 Gluing and reordering the paths P_i

Given the set of 1-edges, we now define a graph G on the set $\mathcal{L} = \{g_{i-1}, h_i, i = 1, \ldots s\}$ ($\#\mathcal{L} \leq s$) of the vertices occurring in 1-edges, using the convention that $g_0 = g_s$. We draw an edge between g_{i-1} and h_i for $i = 1, \ldots, s$. Note that multiple connections are allowed. We denote by $l, 1 \leq l \leq s$, the number of connected components of G. We then group all together the subpaths associated to vertices of the same connected component of G. This leads to l subsets which we call *clusters*. Clusters are ordered in the order they are encountered in P and we denote them by $\mathcal{S}_j$, $1 \leq j \leq l$.

These clusters will now be used to build a new path P' from P as follows. We first define a way to glue the subpaths belonging to the same cluster. For any $1 \leq j \leq l$, we will denote by P_j^g the final path obtained by the gluing of the subpaths from the cluster $\mathcal{S}_j$.

- Assume first that $\sharp\mathcal{L} = s$ so that each 1-edge occurs exactly twice in P. Consider the first cluster which, by definition, begins with the subpath P_1. We first read P_1 until meeting the edge $\binom{h_1}{1}$. If $g_s = h_1$ (then $\mathcal{S}_1$ contains only P_1), then the process stops and the path P_1^g is equal to P_1. Otherwise, there exists $j_o \geq 2$ such that P_{j_o} has the edge $\binom{h_1}{1}$ as endpoint. In the case where $\binom{h_1}{1}$ is the left endpoint of P_{j_o}, we concatenate P_1 and P_{j_o} and then erase the two occurrences of the 1-edge $\binom{h_1}{1}$. In the case where the edge $\binom{h_1}{1}$ is the right endpoint of P_{j_o}, we read P_{j_o} in the reverse order and apply a similar procedure. This "gluing" defines a new subpath which we denote by $P_1 \vee P_{j_o}$. We then restart the procedure with P_1 replaced with $P_1 \vee P_{j_o}$ until all the subpaths belonging to the first cluster are glued leading to the final subpath P_1^g. We then proceed in the same way with other clusters.

- If $\sharp\mathcal{L} < s$ then some clusters have 1-edges that occur four times or more in P. We can find a way to read all the edges of such a cluster without "raising the pen". This follows from the fact that the vertices of G are all of even valency. We then choose one way to do so and glue the paths of these clusters accordingly. For clusters having 1-edges that occur only twice, we apply the previous gluing method.

We end up with l paths $P_j^g, j = 1, \ldots, l$, which begin and end with a 1-edge. By definition of the clusters, these 1-edges form l pairwise distinct pairs of oriented edges. We then call P' the path obtained by the concatenation of the P_j^g 's that is $P' = P_1^g \cup \ldots \cup P_l^g$. The length of P' is $2(s_N - (s - l))$ and its origin is 1, which is a non marked vertex on the bottom line. We call x' its trajectory.

The basic idea of the gluing procedure defined above can be roughly explained as follows. A path P (or P') is said to be typical if it contributes in a non-negligible way to the expectation (12). We first identify the typical paths P'. The simplest of these typical paths are such that the number of occurrences of 1-edges is determined by the number of returns to 0 of their associated trajectory x'. Then, given a typical path P', one has to estimate the number of paths P that can be associated to it as well as their expectation. When considering the expectation, we shall take into account the added weight due to the erased 1-edges. This problem will be considered in the following section. We here establish the needed estimate for the number of preimages P of a path P'.

2.3.3 Number of preimages of a glued path P'

The simplest case is when $l = s$ since all the preimages P coincide with P' up to the translation of the origin. Then if the first return to 0 of the trajectory x' associated to P' holds at the instant $T = 2s_1$, there are exactly (resp. at most) s_1 preimages of P' if x' returns $m = s$ (resp. $m < s$) times to the level 0. In the other case where $l < s$, the following estimate holds true.

Lemma 2.1. *Assume that the first return to 0 of x' holds at time $T = 2s_1$. Then the number of preimages P of the path P' does not exceed*

$$ s_1 \binom{s}{l} (2s_N)^{s-l} . \tag{22} $$

Proof of Lemma 2.1: In order to reconstruct the initial subpaths $P_i, i = 1, \ldots, s$ from P', we first need to choose the $s - l$ instants where we have erased 1-edges. The set $\mathcal{T}$ of these instants combined with the l occurrences of pairs of 1-edges in P' (which determine the paths P_j^g) define s subpaths P_i' which are the subpaths P_i possibly read in the reverse direction. Then one has to define the order in which the subpaths P_i are read. The sole constraint on this order bears on the path starting each cluster, as we now explain. Consider for instance a cluster, say $\mathcal{S}_j$ in P and its corresponding counterpart P_j^g in P'. Call t the first instant of $\mathcal{T}$ chosen in P_j^g. Then the subpath P_i' starting P_j^g and ending at t is the first subpath (with the same direction) of the cluster $\mathcal{S}_j$ met in P. Last and in order to define completely the path P, one also has to choose the origin of P. Thanks to the above, we can now show that the number of preimages of a given path P' does not exceed:

$$ s_1 \binom{s_N - 1}{s - l} \frac{s!}{l!} 2^{s-l} . \tag{23} $$

It is clear that the previous binomial coefficient comes from the choice of the $s - l$ instants of $\mathcal{T}$ (noticing that these instants are necessarily odd). To explain the remaining terms in (23), we denote by x_j the number of subpaths P_i in each cluster $\mathcal{S}_j, 1 \leq j \leq l$. Then, let us consider the first cluster $\mathcal{S}_1$. As clusters are interlaced in P, we also have to choose the places where we read the $x_1 - 1$ paths of $\mathcal{S}_1$ (different from P_1) and choose the order in which we read them. There are $\binom{s-1}{x_1-1}(x_1 - 1)!$ such choices. Furthermore one can also choose the direction in which one reads each of the P_i' (to obtain P_i) not beginning P_1^g. There are two choices for this direction. Having done so, the first empty "slot" corresponds necessarily to the time where we read the first path of the second cluster. We use the same procedure to define the order and the direction in which the remaining subpaths of the second (and subsequent) clusters are read. Thus the number of ways to determine and reorder the subpaths P_i is at most

$$\binom{s_N - 1}{s - l} \prod_{j=1}^{l} \binom{s - 1 - \sum_{j'=0}^{j-1} x_{j'}}{x_j - 1} (x_j - 1)! 2^{s-l} \leq \binom{s_N - 1}{s - l} \frac{s!}{l!} 2^{s-l},$$

where we took the convention that $x_0 = 0$. Last, we shall add a term s_1 coming from the determination of the origin i_0 of the initial path P (which amounts to choosing a vertex occurring on the bottom line of P_1). This yields (23) and it is then easy to deduce Lemma 2.1. $\square$

3 Estimate of $\mathbb{E}\left[\mathrm{Tr}V_N^{s_N}\right]$ when $\Sigma = \mathrm{diag}(\pi_1, 1, \ldots, 1)$

We here prove the universality of the expectation (12) in various scales s_N depending on the value of π_1 with respect to the critical value $w_c = 1 + 1/\sqrt{\gamma}$. Let $c > 0$ be a given real number. In the next theorem, (s_N) is a sequence of integers such that

$$\begin{cases} \lim_{N \to \infty} \frac{s_N}{\sqrt{N}} = c & \text{if } \pi_1 > w_c, \\ \lim_{N \to \infty} \frac{s_N}{N^{2/3}} = c & \text{if } \pi_1 \leq w_c. \end{cases}$$

Theorem 3.1. *Let V_N be a complex (resp. real) matrix satisfying $(\mathrm{H}_1) - (\mathrm{H}_3)$. If $\pi_1 \geq w_c$, we also assume that V_N satisfies (H_4) (resp. (H_4')).*

(i) *Assume that $\pi_1 > w_c$. Then there exists a constant $\hat{C}_4 > 0$ which depends on $\max_j \mathbb{E}(|X_{1j}|^4)$ such that for N large enough,*

$$\mathbb{E}\left[\mathrm{Tr}\, V_N^{s_N}\right] \leq \hat{C}_4 \tau(\pi_1)^{s_N} \quad \text{and} \quad \mathbb{E}\left[\mathrm{Tr}\, V_N^{s_N}\right] = \mathbb{E}\left[\mathrm{Tr}\,(V_N^G)^{s_N}\right](1 + o(1)).$$

(ii) *Assume that $\pi_1 = w_c$. Then there exists a constant $\hat{C}_4' > 0$ which depends on $\max_j \mathbb{E}(|X_{1j}|^4)$ such that for N large enough,*

$$\mathbb{E}\left[\mathrm{Tr}\, V_N^{s_N}\right] \leq \hat{C}_4' u_+^{s_N} \quad \text{and} \quad \mathbb{E}\left[\mathrm{Tr}\, V_N^{s_N}\right] = \mathbb{E}\left[\mathrm{Tr}\,(V_N^G)^{s_N}\right](1 + o(1)).$$

(iii) Assume that $\pi_1 < w_c$. Then there exists a constant $C > 0$ such that for N large enough,

$$\mathbb{E}\big[\mathrm{Tr}\, V_N^{s_N}\big] \leq C u_+^{s_N} \quad \text{and} \quad \mathbb{E}\big[\mathrm{Tr}\, V_N^{s_N}\big] = \mathbb{E}\big[\mathrm{Tr}\, (M_N^G)^{s_N}\big](1 + o(1)).$$

More precisely, in *(i)* if V_N is complex, one can show that $\mathbb{E}\left[\mathrm{Tr}V_N^{s_N}\right] = (1+o(1))\tau(\pi_1)^{s_N} \exp\left[(s_N^2/2N)\left(\frac{\sigma(\pi_1)}{\tau(\pi_1)}\right)^2\right]$. In the real setting, the same estimate holds with $\sigma(\pi_1)$ replaced by $\sqrt{2}\sigma(\pi_1)$. These estimates can trivially be deduced from Theorem 1.2 *(i)* and its real counterpart (due to [18]) combined with some considerations close to those made in Section 2 of [11].
We point out that in the three regimes, the asymptotics of $\mathbb{E}\left[\mathrm{Tr}V_N^{s_N}\right]$ differ in the complex and real settings. This is not surprising since the limiting distributions of the largest eigenvalues are different. Through the combinatorial analysis, this fact is justified by the existence of non-MP-closed vertices in some typical paths. The investigation of such vertices is here really similar to that made in [20] and we refer to Section 2.2 above and [20] for more detail.

This section is devoted to the proof of Theorem 3.1. The contribution to the expectation (12) of paths with no 1-edges has been evaluated in Proposition 2.2. This section is devoted to the estimation of the contribution from edge paths P having 1-edges. We shall show that if $\pi_1 < w_c$, this contribution is negligible with respect to that from paths without 1-edges (which is of the order of $\mathbb{E}\left[\mathrm{Tr}M_N^{s_N}\right]$, cf. Section 2.2 above). On the other hand, when $\pi_1 \geq w_c$, we shall prove that paths with 1-edges contribute in a non-negligible way. As announced, our proof will make use of the gluing procedure. Thus, we will first consider the glued paths P' and find the typical ones that is those which contribute in a non-negligible way to the expectation. We will easily see that the typical paths P' have all their edges passed twice. Then, given a typical path P', we shall estimate the contribution of all its preimages P. This will require to examine the added weight due to the erased 1-edges: by construction of the gluing procedure, it may happen that P has some 1-edges passed at least four times. We shall check that the contribution of the typical paths P depends only on the second and fourth moments of the X_{ij}'s.

Before we proceed, we need a few notations. A glued path P' is of length $2(s_N - (s - l))$ where s (resp. l) denotes the number of pairs of 1-edges (resp. of clusters) in its preimages P. Note that $s - l$ counts the number of pairs of 1-edges that have been erased by the gluing procedure; l counts the pairs of 1-edges in P'. Besides, the origin of P' is the vertex 1 and is a non marked vertex on the bottom line. Throughout the paper, we also denote by m the number of times the trajectory of P' goes back to the level 0. Note that in general $m \leq l$.

Definition 3.1. *We call $\mathbf{N}(s_N - (s - l), k, m)$ the number of Dyck paths of length $2(s_N - (s - l))$ with k odd marked instants and m returns to 0.*

The simplest case to deal with is when $m = l$, that is when all the occurrences of the vertex 1 on the bottom line of P' are encountered at the instants where its trajectory hits the level 0. Thus, when $m = l$, estimating the contribution to (12) of paths P follows from statistics on the number of returns to 0 of the underlying Dyck paths of P' (each return is weighted by π_1). This observation justifies the following definition.

Definition 3.2. *A path P' is said to be a fundamental path if all its 1-edges occur at level 0.*

In the following subsection, we present the detailed computations of the contribution to (12) from edge paths P associated to a fundamental path P'. In Subsection 3.2, we consider the set of non-fundamental paths P' that is the case where some clusters in the initial paths P do share edges in such a way that some 1-edges in P' occur at levels greater than 0. As we will see, this requires to refine the analysis and define a new gluing procedure.

3.1   All the occurrences of the vertex 1 on the bottom line are made at level 0

In this subsection, we investigate the set of paths P such that the l different clusters $\mathcal{S}_j, j = 1, \ldots, l$, after the gluing procedure, yield a fundamental path P'. We start from such P' and examine the possible added weight when reversing the gluing process, that is the expectation from the erased 1-edges. The simplest case, examined in the subsequent proposition, is when there is no erased 1-edge through the gluing process.

Denote by Z_1 the contribution of paths P having 1-edges passed at most twice and whose associated glued path P' satisfies $m = l = s$. P' has then s returns to 0 and length $2s_N$.

Proposition 3.1. *One has that*

(i) $Z_1 = O(1)\tau(\pi_1)^{s_N}$ *if $\pi_1 > w_c$ and $s_N = O(\sqrt{N})$,*

(ii) $Z_1 = O(1)u_+^{s_N}$ *if $\pi_1 = w_c$ and $s_N = O(N^{2/3})$,*

(iii) $Z_1 = o(1)u_+^{s_N}$ *if $\pi_1 < w_c$ and $s_N = O(N^{2/3})$.*

Proof of Proposition 3.1:   By assumption, the paths P and P' coincide up to a translation of the origin. Furthermore, the vertex 1 is the origin of P', is non marked and, by definition of the gluing procedure, the path P' has 1-edges seen at most twice. To define P from P', one only has to determine the origin of the path P, which amounts to choosing an even instant before the first return to 0 of the trajectory associated to P'.

Let α, α' be such that $0 < \alpha' < \frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} < \alpha < 1$. Call $\hat{Z}_1$ the contribution from paths P associated to fundamental paths and for which the number k of

odd up steps satisfies $\alpha' s_N \leq k \leq \alpha s_N$. The computations of [20] (Section 3), summarized in Section 2.2, can be copied to show that typical paths P' of length $2s_N$ having k odd up steps and s returns to the level 0 have edges passed only twice. Let $\mathbb{E}_{k,s}$ denote the expectation with respect to the uniform distribution on the set of Dyck paths with k odd up steps and s returns to the level 0. It is in particular a minor modification to show that the estimate (20) holds when $\mathbb{E}_k$ is replaced with $\mathbb{E}_{k,s}$ (and $\mathbf{N}(s_N, k)$ with $\mathbf{N}(s_N, k, s)$) uniformly in s : the proof can be deduced for instance using arguments given in [11], Lemma 7.10 (2nd case). As already recalled in Section 2.2, it is also proved in [20] that a typical path of length $2s_N$ with s returns to 0 has an unmarked origin, no edge read more than twice and no vertex of type strictly greater than 3. Thus, we can deduce that typical paths P in $\hat{Z}_1$ have edges passed only twice. From the above, we deduce that

$$\hat{Z}_1 = O(1) \times \sigma^{2s_N} \sum_{1 \leq s_1 \leq s_N} \sum_{1 \leq k_1 \leq s_N} s_1 \mathbf{N}(s_1 - 1, s_1 - k_1) \sum_{m=1}^{s_N - s_1}$$
$$\sum_{k=k_1+m-1}^{k_1+s_N-s_1} \mathbf{N}(s_N - s_1, k - k_1, m - 1) \pi_1^m \gamma_N^{k-s_N}. \tag{24}$$

We now consider the paths P contributing to Z_1 and for which $k \geq \alpha s_N$ or $k \leq \alpha' s_N$ and show that they contribute in a negligible way to Z_1. To this aim, let $\hat{k} := \left[\frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} s_N \right] + 1$. Here we show that there exists a constant $C > 0$ such that for any integer n (with $0 < \hat{k} + n \leq s_N$),

$$\mathbf{N}(s_N, \hat{k} + n, m) \gamma_N^{\hat{k}+n} \leq C e^{-Cn^2/s_N} \mathbf{N}(s_N, \hat{k}, m) \gamma_N^{\hat{k}}. \tag{25}$$

This will imply that the main contribution to Z_1 comes from paths with approximately $\hat{k}$ odd marked instants, so that $Z_1 = \hat{Z}_1(1 + o(1))$. To prove (25), we write

$$\mathbf{N}(s_N, \hat{k} + n, m) \gamma_N^{\hat{k}+n} = \sum_{s_1,\ldots,s_m}^{*} \sum_{k_1,\ldots,k_m}^{*} \prod_{i=1}^{m} \mathbf{N}(s_i - 1, s_i - k_i) \gamma_N^{k_i},$$

where the starred sums bear on integers s_i summing to s_N and k_i summing to $\hat{k} + n$. We also set $\hat{k}_i = \left[\frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} s_i \right] (+1)$ so that $\sum_i \hat{k}_i = \hat{k}$. Using the ideas of Remark 2.4 of [20], one has that

$$\mathbf{N}(s_i - 1, s_i - k_i) \gamma_N^{k_i} \leq e^{\{-C' \frac{(k_i - \hat{k}_i)^2}{s_i - 1}\}} \mathbf{N}(s_i - 1, s_i - \hat{k}_i) \gamma_N^{\hat{k}_i}$$

for some constant $C' > 0$, independently of s_i. Furthermore setting $k_i - \hat{k}_i = x_i(s_i - 1)$, one can easily show that

$$\sum_{k_1,\ldots,k_m}^{*} \exp\left(-C' \sum_{i=1}^{m} \frac{(k_i - \hat{k}_i)^2}{s_i - 1} \right)$$

21

$$= \int \cdot \int \exp\left(-C'\left(\sum_{i=1}^{m-1}(s_i - 1)x_i^2 + \frac{(n - \sum_{i=1}^{m-1}(s_i - 1)x_i)^2}{s_m - 1}\right)\right) \prod_{i=1}^{m-1}(s_i - 1)\, dx_i$$

$$\leq e^{-C\frac{n^2}{s_N}} \prod_{i=1}^{m-1} \sqrt{s_i - 1},$$

for some constant C independent of the s_i's and m. Moreover, one can show that $\mathbf{N}(s_N, \hat{k}, m) \leq C \prod_i \sqrt{s_i - 1}\, \mathbf{N}(s_i - 1, s_i - \hat{k}_i)$. Indeed, given s_i, the number of $\tilde{k}_i$ contributing in a non negligible way to $\sum_{\tilde{k}_i} \prod_{i=1}^{m} \mathbf{N}(s_i - 1, s_i - \tilde{k}_i)\gamma_N^{\tilde{k}_i}$, where $\sum_i \tilde{k}_i = \hat{k}$, is of the order $\sqrt{s_i - 1}$ and each product term is of order of $\prod_{i=1}^{m} \mathbf{N}(s_i - 1, s_i - \hat{k}_i)\gamma_N^{\hat{k}_i}$ (uniformly in s_i). Thus we get (25) and we can conclude directly that the contribution of paths for which $k \geq \alpha s_N$ or $k \leq \alpha' s_N$ is negligible and that $Z_1 = \hat{Z}_1(1 + o(1))$.

There now remains to prove that (24) yields Proposition 3.1. This is obtained from Lemma 3.1 stated and proved below. $\square$

We now turn to the proof of the announced Lemma 3.1. Let $n \geq 1$ be an integer. Set

$$a_n = \sum_{s_1=1}^{n} \sum_{k_1=1}^{n} s_1 \mathbf{N}(s_1 - 1, s_1 - k_1) \sum_{s=1}^{n-s_1} \sum_{k=k_1+s-1}^{k_1+n-s_1} \mathbf{N}(n - s_1, k - k_1, s - 1) \frac{\pi_1^s \gamma_N^k}{\gamma_N^n},$$

(26)

so that $(24) = O(1) \times \sigma^{2s_N} a_{s_N}$.

Lemma 3.1. Let $a'_n = \sigma^{2n} a_n$. For n large enough, one has that
(i) if $\pi_1 > w_c$ then $a'_n = \tau(\pi_1)^n(1 + o(1))$; (ii) if $\pi_1 = w_c$ then $a'_n = u_+^n(1 + o(1))$; (iii) if $\pi_1 < w_c$ then $a'_n = \frac{1}{\sqrt{n}} u_+^n(1 + o(1))$.

Proof of Lemma 3.1: The proof makes use of various generating functions, for which we need a few definitions. Let χ_n denote the set of Dyck paths of length $2n$. For a trajectory $x \in \chi_n$, we define

$$r_x := \sharp\{t \in]0, 2n[, \ x(t) = 0\}, \ o_x := \sharp\{ \text{ odd marked instants of } x \}$$
$$e_x := \sharp\{ \text{ even marked instants of } x \}.$$

Introduce the generating functions

$$F(\pi_1, \gamma, z) = \pi_1 \sum_{n\geq 0} \sum_{x \in \chi_n} \pi_1^{r_x} \gamma^{-e_x} z^n, \ K(z) := \sum_{n\geq 0}(n+1) \sum_{x\in\chi_n} \gamma^{-o_x} z^{n+1}.$$

Then the function

$$H(z) := F(\pi_1, \gamma, z)K(z) \tag{27}$$

is "almost" the generating function associated to the terms a_n and thus to those occurring in (24). Indeed in the definition of a_n as in (24), we have γ_N instead of γ. This will have no impact on the following reasoning as $\lim_{N\to\infty} \gamma_N = \gamma$ and we are interested in large N-asymptotics. Thus expanding H as a power series $H(z) := \sum_{n=0}^{\infty} a_n z^n$, one has that $Z_1 = O(1) \times a'_{s_N}$.

In order to determine the asymptotics of a_n, we now turn to the evaluation of the generating functions. This is the aim of the subsequent lemma.

Lemma 3.2. *One has that*

$$F(\pi_1, \gamma, z) = \frac{2}{2/\pi_1 - (1 - \gamma^{-1})z - 1 + \sqrt{(1 + (1 - \gamma^{-1})z)^2 - 4z}}$$

and

$$K(z) = \frac{z}{2}\frac{\partial}{\partial z}\left((1 - \gamma^{-1})z + 1 - \sqrt{(1 + (1 - \gamma^{-1})z)^2 - 4z}\right).$$

Proof of Lemma 3.2: We need to define two auxiliary generating functions to prove Lemma 3.2. Set

$$G(\gamma, z) = \sum_{n \geq 0}\sum_{x \in \chi_n} \gamma^{-o_x} z^n, \quad \tilde{G}(\gamma, z) = \sum_{n \geq 0}\sum_{x \in \chi_n} \gamma^{-e_x} z^n.$$

Then, decomposing any trajectory $x \in \chi_n$ when $n > 0$ according to the first return to the origin, one deduces the following relations:

$$\tilde{G}(\gamma, z) = 1 + zG(\gamma, z)\tilde{G}(\gamma, z), \quad G(\gamma, z) = 1 + \gamma^{-1}z\tilde{G}(\gamma, z)G(\gamma, z),$$
$$F(\pi_1, \gamma, z) = \pi_1 + \pi_1 z G(\gamma, z)F(\pi_1, \gamma, z). \tag{28}$$

Solving these equations yields (see [27]) that $F(\pi_1, \gamma, z) = \dfrac{\pi_1}{1 - \pi_1 z G(\gamma, z)}$, where

$$G(\gamma, z) = \frac{(1 - \gamma^{-1})z + 1 - \sqrt{(1 + (1 - \gamma^{-1})z)^2 - 4z}}{2z}.$$

For the evaluation of K, it is enough to observe that

$$
\begin{aligned}
K(z) &= z\sum_{n \geq 0}(n + 1)\sum_{x \in \chi_n} \gamma^{-o_x} z^n = z\frac{\partial}{\partial z}\sum_{n \geq 0}\sum_{x \in \chi_n} \gamma^{-o_x} z^{n+1} \\
&= z\frac{\partial}{\partial z}\left(zG(\gamma, z)\right).
\end{aligned} \tag{29}
$$

This finally yields Lemma 3.2. $\square$

Thanks to Lemma 3.2, one shall then deduce the asymptotics of a_n as n goes to infinity from the generating function $H(z) := F(\pi_1, \gamma, z)K(z)$. Set $U = zG(\gamma, z)$. It can be deduced from [27] (pp. 21-22) that U is holomorphic in the disk $\{z, |z| < \sigma^2/u_+\}$, and one has that $z = \frac{U(U-1)}{(1-\gamma^{-1})U-1}$. Furthermore, $z = 0$ if $U = 0$. Assume first that $\pi_1 \leq w_c$. One thus has that

$$a_n := a_n'/\sigma^{2n} = \frac{1}{2i\pi}\int_{\mathcal{C}_o}\frac{1}{z^n}\frac{\frac{d}{dz}(zG(\gamma, z))}{1/\pi_1 - zG(\gamma, z)}\,dz,$$

23

where the contour $\mathcal{C}_o$ encircles 0, is oriented counterclockwise and lies in the disk $\{z, |z| < \sigma^2/u_+\}$. By a straightforward change of variables, one gets that

$$a_n = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{1}{1/\pi_1 - u} \left(\frac{(1 - \gamma^{-1})u - 1}{u(u-1)} \right)^n du,$$

where $\mathcal{C}$ is a symmetric contour encircling 0 and remaining on the left of $\sigma/\sqrt{u_+}$. Note that this implies that the contour $\mathcal{C}$ cannot encircle $1/\pi_1$. It is then an easy saddle point argument to check points (ii) and (iii): the critical point is $u_c := 1/w_c$ and the saddle point contour is modified in a neighborhood of width $1/\sqrt{n}$ of u_c so that $\mathcal{C}$ remains to the left of u_c.
When $\pi_1 > w_c$, the contour $\mathcal{C}$ does not encircle $1/\pi_1$ and a straightforward Laplace method leads to point (i). This finishes the proof of Lemma 3.1. $\square$

We now turn to estimating the contribution of paths P with 1-edges seen only twice and which give a fundamental glued path P' by erasing a positive number of 1-edges (precisely $2(s - l)$ with our notations). We call Z_2 this contribution. Note that the glued paths P' to be considered here are such that $m = l$ with $m < s$.

Proposition 3.2. *There exists a constant $C > 0$ such that $Z_2 \leq CZ_1$.*

The proof will make use of the following extension of Lemma 3.1. In the next lemma, we write $a'_n = a'_n[\pi_1]$ (recall that $a'_n = \sigma^{2n} a_n$ with a_n given by (26)).

Lemma 3.3. *Assume that $\pi_1 \leq w_c$. Let $C > 0$ be some constant independent of N. Then for all large N, and as long as $s_N = O(N^{2/3})$, there exists a constant C' depending on C only such that $a'_{s_N}\left[\pi_1 e^{Cs_N/N}\right] = a'_{s_N}[\pi_1](C' + o(1))$.*

We skip the proof of this lemma which can be obtained by the same saddle point argument as in Lemma 3.1. We now turn to the proof of Proposition 3.2.

Proof of Proposition 3.2: Let P' be a fundamental path. It can first be deduced from Lemma 2.1 (applied with $l = m$) that the number of paths P which are preimages of P' and contribute to Z_2 is at most

$$s_1 \frac{(2ss_N)^{s-m}}{(s-m)!} \tag{30}$$

where s_1 is such that the first return to zero of P' occurs at time $2s_1$. We already know from [20] and [26] that in typical paths P' no edge is read more than twice. As it is assumed that P has no 1-edges seen 4 times or more, the expectation of P is just $(\pi_1\sigma^2/p)^{s-m}$ times that of P'. Using the inequality $\mathbf{N}(s_N - (s-m), k, m-1) \leq \mathbf{N}(s_N, k+s-m, s-1)$ and setting $s' = s - m \geq 1$ and $k' = k + s'$, one can check that there exists a constant $C > 0$ (whose value may vary from line to line) such that

$$Z_2 \leq C\sigma^{2s_N} \sum_{s_1=1}^{s_N} \sum_{s=1}^{s_N-s_1} \sum_{m=1}^{s-1} \sum_{k=1}^{s_N} \sum_{k_1 \leq k'} s_1 \mathbf{N}(s_1 - 1, s_1 - k_1)$$

24

$$\mathbf{N}(s_N - (s-m) - s_1, k - k_1, m - 1)\frac{\left(\frac{2ss_N}{p}\right)^{s-m}}{(s-m)!}\, \pi_1^s \gamma_N^{k-s_N} \quad (31)$$

$$\leq \quad C\sigma^{2s_N} \sum_{s_1=1}^{s_N} \sum_{s=1}^{s_N-s_1} \sum_{s'=1}^{s_N} \sum_{k'=s'+1}^{s_N} \sum_{k_1 \leq k'} s_1 \mathbf{N}(s_1 - 1, s_1 - k_1)$$

$$\mathbf{N}(s_N - s_1, k' - k_1, s - 1)\frac{1}{s'!}\left(\frac{2\gamma_N^{-2}ss_N}{N}\right)^{s'} \pi_1^s \gamma_N^{k'-s_N}$$

$$\leq \quad C\sigma^{2s_N} \sum_{s_1=1}^{s_N} \sum_{s=1}^{s_N-s_1} \sum_{k'=1}^{s_N} \sum_{k_1 \leq k'} s_1 \mathbf{N}(s_1 - 1, s_1 - k_1)$$

$$\mathbf{N}(s_N - s_1, k' - k_1, s - 1)\big\{\exp\left(Css_N/N\right) - 1\big\}\pi_1^s \gamma_N^{k'-s_N}. \quad (32)$$

In (31), the factor $(2ss_N/p)^{s'}$ can be deduced from (30) and the fact that the rescaling factor p^{s_N} splits into $p^{s_N-s'}p^{s'}$. In the case where $\pi_1 > w_c$ and $s_N = O(\sqrt{N})$, we then readily get the result. For the case where $\pi_1 \leq w_c$ and $s_N = O(N^{2/3})$, the conclusion follows from Lemma 3.3. $\square$

Amongst the paths P associated to a fundamental glued path P', there remains to consider those with some 1-edges seen four times or more. By definition of the gluing procedure, these edges must be erased and appear at most twice in P'. Thus, as for Z_2, one has that $m = l$ and $s > m$ (and there are at most $(s-m)$ 1-edges seen 4 times or more in P). We call Z_3 the contribution of these paths and we show that those contributing to Z_3 in a non-negligible way have 1-edges passed at most 4 times.

Proposition 3.3. *There exists a constant $\hat{C}_4 > 0$ which depends on the fourth moments of the entries of X such that $Z_3 \leq \hat{C}_4 Z_2$.*

Observe that Z_3 is non-negligible when $\pi_1 \geq w_c$ which partly explains the added constraint on the fourth moments of the X_{ij}'s to get the announced universality in cases $\pi_1 > w_c$ and $\pi_1 = w_c$.

Proof of Proposition 3.3: One already knows from [20] and [26] that in typical paths P' no edge is read more than twice. Yet, in paths contributing to Z_3, there exist some vertices that occur more than twice on each of the top and bottom lines. Thus choosing $s - m$ moments of time in P' (to reconstruct P) can result into a 1-edge which is read more than twice in P. Note that by definition of clusters, such an edge can only be read inside one cluster in P. We now estimate the expectation of the path P with respect to that of P'. We call $\hat{Z}_3$ the contribution to Z_3 from paths P with 1-edges read 4 times at most and Z_3^0 denotes the remaining contribution to Z_3. We show that $\hat{Z}_3$ is of the order of Z_2 while $Z_3^0 = o(1)Z_2$. Our reasoning is mainly based on several properties of the gluing procedure which has been defined in Section 2.3 above.

Consider a 1-edge $e = \begin{pmatrix} v \\ 1 \end{pmatrix}$ which is read at least 4 times in P. Assume it is

read $2y_1 \geq 4$ times in P which, using (H$_2$), implies that its expectation is at most $(Cy_1)^{y_1}$ for some constant $C > 0$ independent of N.

1^{rst} *case:* The edge e does not coincide with any of the edges of P'. In other words, e is distinct from the edges starting the different clusters in P'. This means that amongst the $s-m$ instants in P' where we have erased 1-edges (that is the set $\mathcal{T}$ in Section 2.3), we have chosen y_1 times edges with vertex v on the top line. These choices split the subpath P_1^g (derived from the cluster $\mathcal{S}_1$ by the gluing procedure) into $y_1 + 1$ subpaths as follows:

$$\begin{pmatrix} g_s \\ 1 \end{pmatrix} \cdots \begin{pmatrix} v \\ \alpha_1 \end{pmatrix} \begin{pmatrix} v \\ \beta_1 \end{pmatrix} \cdots \begin{pmatrix} v \\ \alpha_2 \end{pmatrix} \begin{pmatrix} v \\ \beta_2 \end{pmatrix} \cdots \begin{pmatrix} v \\ \alpha_{y_1} \end{pmatrix} \begin{pmatrix} v \\ \beta_{y_1} \end{pmatrix} \cdots \begin{pmatrix} g_s \\ 1 \end{pmatrix}. \tag{33}$$

Denote now by $\tilde{Q}_j, j = 1, \ldots, y_1 - 1$, the subpath starting with the edge $\begin{pmatrix} v \\ \beta_j \end{pmatrix}$ and ending with the edge $\begin{pmatrix} v \\ \alpha_{j+1} \end{pmatrix}$. Let also $\tilde{Q}_o$ (resp. $\tilde{Q}_f$) be the subpath starting with $\begin{pmatrix} g_s \\ 1 \end{pmatrix}$ (resp. $\begin{pmatrix} v \\ \beta_{y_1} \end{pmatrix}$) and ending with $\begin{pmatrix} v \\ \alpha_1 \end{pmatrix}$ (resp. $\begin{pmatrix} g_s \\ 1 \end{pmatrix}$). Then each path $\tilde{P}'$ which is obtained from P' by permuting any of the $\tilde{Q}_j, j = 1, \ldots, y_1 - 1$ leads by permuting the paths P_i to the same path P. Thus, the number of preimages of such a path P' has to be divided by a factor $(y_1 - 1)!$ since each preimage is counted $(y_1 - 1)!$ times when considering all the possible paths P' (recall the proof of Lemma 2.1). Taking into account the expectation of the edge e in P then adds a factor $(Cy_1)^{y_1}/(y_1 - 1)! \leq C^{y_1}$.

Let us count now the number of ways to select $s-m$ moments of time in such a way that we define y_1 times the same edge e. For this, assume that the instant $\tilde{t}$ where $\tilde{Q}_1$ begins in P' has been chosen. Then two situations may happen when choosing the instant $\tilde{t}'$ where it ends. To explain this, we need to introduce two characteristics (already mentioned in Section 2.2) of the path P'. The first one is $\nu_N = \nu_N(P')$, the maximal number of vertices that can be visited in P' at marked instants from a given vertex different from the origin 1. The second one is $T_N = T_N(P')$, the maximal type of a vertex in P'. We shall use the following fact deduced from the very definitions of ν_N and T_N : given a vertex (different from the origin 1) occuring in P', it appears at most $T_N + \nu_N$ (resp. T_N) times as endpoint (resp. right endpoint) of up steps. It is then not hard to see that a given vertex (distinct of the origin 1) appears at most $2(T_N + \nu_N)$ times along the path P'. Thus, the number of ways to select y_1 times the same vertex v when choosing (in P') the $s-m$ moments of time does not exceed:

$$\begin{pmatrix} s_N \\ s - m - y_1 \end{pmatrix} \times (s_N - s + m + y_1) \times \begin{pmatrix} 2(T_N + \nu_N) \\ y_1 - 1 \end{pmatrix}$$
$$\leq C^{s-m} \begin{pmatrix} s_N \\ s - m \end{pmatrix} \left(\frac{2(T_N + \nu_N)}{s_N - s + m} \right)^{y_1 - 1},$$

for some constant $C > 0$.

26

2nd *case:* The edge e coincides with one of the edges of P'. In this case, the above reasoning on the permutation of the subpaths $\tilde{Q}_j$ still applies. One then needs to distinguish two cases according to the value of y_1.

Case (a): $y_1 = 2$. Then the edge e is seen exactly four times in P. To determine e one has to select one of the m edges starting a cluster. This determines the vertex v. The occurrence of v along the path P' where we have erased e has then to be determined. In principle, there are at most $\nu_N + T_N$ possible choices for this occurrence. Nevertheless it is the most probable that $m(T_N + \nu_N) >> s_N$ due to the fact that m may be large. Thus, calling on the characteristics ν_N and T_N does not improve the estimate and it is sufficient to notice, as in Lemma 2.1, that the number of ways to choose the $s - m$ moments of times to determine the erased 1-edges is at most of order

$$\binom{s_N}{s - m},$$

that is exactly as for Z_2.

Case (b): $y_1 > 2$. In this case, once v is determined (with at most s_N ways to do so), there are at most $\binom{2(T_N + \nu_N)}{y_1 - 2}$ possible ways to select $y_1 - 2$ other repetitions of v in P'. Thus, the number of ways to select the $s - m$ moments of time to determine the 1-edges in this case does not exceed

$$C^{s-m} \binom{s_N}{s - m} \left(\frac{2(T_N + \nu_N)}{s_N - s + m} \right)^{y_1 - 2}.$$

We can now conclude that Z_3^0 is negligible. Indeed, we deduce that

$$Z_3^0 \leq 2 Z_2 \sum_{y_1 \geq 3} \left(\frac{2C'(T_N + \nu_N)}{s_N - s + m} \right)^{y_1 - 2} = o(1) Z_2, \qquad (34)$$

since for typical paths P', one can show that $s_N - s + m \to \infty$ (using (31) and the above) and that $(T_N + \nu_N)^2 << s_N - s + m$ (this point, recalled in Section 2.2 follows from [20], Section 3.2). Similarly we can also show that the contribution of paths P with 1-edges seen 4 times in P but that do not arise in P' (this corresponds to the situation of the previous 1^{rst} *case* with $y_1 = 2$) is negligible with respect to Z_2 and does not contribute to the expectation (12). Last, it is not hard to see from Case (a) that $\hat{Z}_3$ is of the order of Z_2. More precisely, for each edge e seen four times in P and twice in P', one has to multiply the expectation of P' by $\pi_1 \mathbb{E}|X_{1v}|^4/(\sigma^2 p)$ at most to get that of P. In the sequel we set $\hat{C}_4 := \max_v \mathbb{E}|X_{1v}|^4/\sigma^2 + 1$. As $s - m$ counts the number of instants in P where a 1-edge has been erased, one has that (compare with (31))

$$\hat{Z}_3 \leq C \sigma^{2s_N} \sum_{s_1=1}^{s_N} \sum_{s=1}^{s_N - s_1} \sum_{m=1}^{s-1} \sum_{k=1}^{s_N} \sum_{k_1 \leq k + s - m} s_1 \mathbf{N}(s_1 - 1, s_1 - k_1)$$

$$\mathbf{N}(s_N - (s - m) - s_1, k - k_1, m - 1) \frac{\left(\frac{2\hat{C}_4 s s_N}{p} \right)^{s-m}}{(s - m)!} \pi_1^s \frac{\gamma_N^k}{\gamma_N^{s_N}}. \qquad (35)$$

27

We then conclude (using Lemma 3.3 in the case where $\pi_1 \leq w_c$) that there is another constant $\hat{C}'_4$ depending on the fourth moments of the X_{ij}'s such that $\hat{Z}_3 \leq \hat{C}'_4 Z_2$. This finishes the proof that typical paths P may have edges seen 4 times but not more. Note that this happens when $\pi_1 \geq w_c$ only, and in this case their associated path P' has no (1-)edge seen more than twice. $\square$

3.2 Edges shared by clusters

In this section we investigate paths P such that the clusters do share some edges in such a way that the 1-edges in the glued path P' are not necessarily read at moments of time where the trajectory goes back to the level 0. Keeping the same notations as before, the length of the path P' is now $2(s_N - (s - l))$ and $l - m$ returns to the vertex 1 on the bottom line of P' occur at some positive levels. To consider such paths, we define a second gluing procedure and associate a second path P'' to the initial path P. This gluing procedure is close to the *construction procedure* already used in [25] and [26].

For short, we call Q_i instead of P_i^g the subpaths in-between two returns to the vertex 1 on the bottom line of P' (recall Section 2.3). We let $i_1 \leq l - m - 1$ be the smallest index where the first return to the vertex 1 on the bottom line occurs at some positive level. Then there exists an edge which is opened but not closed in Q_{i_1}. We denote by $\hat{e} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ the first of these edges. When reading P', let then i_2 be the lowest index such that the edge $\hat{e}$ is closed (and odd) in Q_{i_2}. Let then e be the first edge in Q_{i_1} occuring also in Q_{i_2}. Note that it may happen that $e \neq \hat{e}$: this arises in non typical paths only, as this implies that P' has edges seen at least 4 times. Let also t_e and t'_e be the instants of the first occurrence of the edge e in Q_{i_1} and Q_{i_2} respectively. We then define the path $Q_{i_1} \vee Q_{i_2}$ obtained by the gluing of the two subpaths by erasing the first occurrence of the common edge e in each of the subpaths as follows. We first read Q_{i_1} until the left endpoint of the edge e at time t_e. Then we switch to Q_{i_2} in the following way. If t_e and t'_e are of the same parity, we then read Q_{i_2}, starting from t'_e, in the reverse direction to the origin and restart from the end of Q_{i_2} until we come back to the instant $t'_e + 1$. If t_e and t'_e are not of the same parity, we read the edges of Q_{i_2} in the usual direction starting from $t'_e + 1$ and until we come back to the instant t'_e. We have then read all the edges of Q_{i_2} except the edge e occurring between t'_e and $t'_e + 1$. We then read the end of Q_{i_1}, starting from $t_e + 1$. Having done so, we obtain a path $Q_{i_1} \vee Q_{i_2}$ which has the same final (and first) edge as Q_{i_1} and the vertex 1 is marked once on the bottom line. We then set P'_1 to be the path defined by

$$P'_1 = Q_1 \cup \ldots \cup Q_{i_1 - 1} \cup Q_{i_1} \vee Q_{i_2} \cup \hat{Q}_{i_2} \ldots \cup Q_{i_m}.$$

Here the hat means that the corresponding term does not appear. We then replace P' with P'_1 and restart the same procedure. We call $1 \leq g \leq l - m$ the number of gluings needed so that all the occurrences of 1-edges correspond

either to a marked instant or to a return to 0 of the associated trajectory. Note that g is defined by $l - m - g = \sharp \mathcal{D}$ where $\mathcal{D}$ is the set of the subpaths Q_i which are sub-Dyck paths of origin 1 with all their edges even and which occur at some positive levels in all the successive P_i'''s. Note that through the gluing process, such subpaths are not modified but are moved to the level 0 in the order they appear. We denote by P'' the path finally obtained after g steps of the gluing procedure. By definition of this gluing procedure, P'' is of length $2(s_N - (s - l) - g)$ with exactly $m' = l - g$ returns to the level 0, its origin is the vertex 1 and is marked $l - m'(= g)$ times on the bottom line. In the following, we denote by k the number of odd marked instants in P''.

We shall now estimate the number of preimages P of such a path P'' as well as their expectation. The first and main work here is to investigate the step from P'' to P'. Once this is done, it will be quite straightforward to estimate the number of preimages P of such a path P' and their expectation by extending the analysis made in the previous subsection. To reconstruct P' from P'', we have to "recover" each of the g glued subpaths Q_{i_2}. Thus, for each glued subpath $Q_{i_1} \vee Q_{i_2}$, we have to find the instants where Q_{i_2} begins and ends (that is the two instants of switch from one path to the other); we also need to determine the direction in which Q_{i_2} is read as well as the origin of Q_{i_2} in P''. Actually, the origin of Q_{i_2} is just given by the marked occurrence of the vertex 1 in $Q_{i_1} \vee Q_{i_2}$. Then, we shall take into account the weighted contribution to the expectation of each erased edge e. More precisely, one already knows that in typical paths P'', each edge appears only twice. But when rebuilding P' from P'', it may happen that some of the erased edges e appear more than twice in P'. As we will see, such paths will lead to a negligible contribution to the expectation (12), which will ensure the universality.

Consider paths P having some non disjoint clusters so that some 1-edges in the associated glued path P' occur at positive levels. Denote by Z_4 the contribution to the expectation (12) from such paths P for which all the erased edges e between P' and P'' appear exactly twice in P (or P').

Proposition 3.4. *The main contribution to Z_4 comes from paths P with all edges seen twice except 1-edges which possibly occur 4 times. And there exists a constant $\hat{C}_4 > 0$ depending on the fourth moments of the entries of X such that $Z_4 \leq \hat{C}_4 Z_1$.*

Proof of Proposition 3.4: Here we only consider paths P'' having all their edges seen twice since, as previously said, these are the typical paths.

Let us first reconstruct P' from P''. Due to the fact that the vertex 1 is marked $g = l - m'$ times on the bottom line of P'', the weighted number of such paths P'' is at most of order

$$\binom{s_N - (s - l) - g - k}{g} (C/N)^g$$

29

times the weighted number of paths where the origin 1 is non marked and with the same length, the same number of odd marked instants and the same number of returns to 0. Then, by definition of the gluing procedure, the first step beginning the subpath Q_{i_2} in $Q_{i_1} \vee Q_{i_2}$ is up which implies that the total number of ways to choose the instants of initial switch is at most of $\binom{s_N-(s-l)}{g}$. Moreover, the number of possible choices for the instants where one switches for the second time from the subpaths Q_{i_2} to the Q_{i_1} is at most $\binom{s_N-(s-l)-g}{g}g!$ (the factor $g!$ comes from the fact that the Q_{i_2} may be interlaced). It remains to choose the direction and the order in which the subpaths Q_i are read. We claim that this yields a factor $2^g\binom{m'+g}{g} \times g!$. Indeed, one can notice that once the Q_{i_2}'s are identified, the remaining $m'+g$ subpaths are known i.e. one knows the Q_{i_1}'s and the paths belonging to the set $\mathcal{D}$. Moreover, by construction of the gluing process, these latter subpaths appear in the same relative order as in P'. To reorder the Q_i's, one needs first to choose the place where one actually encounters the (unordered) subpaths that are glued and moved by the gluing process i.e. the Q_{i_2} (note that this also reorders those belonging to the set $\mathcal{D}$). There are at most $\binom{m'+g}{g}$ ways to do this. Last, the previous term $g!$ counts the number of ways to reorder the subpaths Q_{i_2}. Regarding the respective weights of the paths P'' and P', one has to take into account the erased edges. As we assume that the erased edges are pairwise distinct and read exactly twice in P, the weight of P' is of order N^{-g} that of P''.

Now, one has to reconstruct P from P'. In fact, this is really close to the analysis made in the previous Section 3.1. Indeed, the upper bound on the number of preimages P of a path P' obtained in Lemma 2.1 does not use the assumption that clusters are disjoint or not. Thus we deduce from Lemma 2.1 that the number of preimages P of a path P' is at most $s_1\binom{s}{l}(2s_N)^{s-l}$, if the first return to 0 of the trajectory of P' holds at time $2s_1$.

We are now in position to estimate the contribution Z_4. Observe first that $\mathbf{N}(s_N-s_1-(s-l)-g,k-k_1,m'-1) \leq \mathbf{N}(s_N-s_1,k+(s-l)+g-k_1,s-1)$ (recall that $m'+s-l+g=s$). Hence, letting $k'=k+(s-l)+g$ and using the fact that $m'=l-g$, by computations similar to those made for the Z_i, $i=1,2,3$ in the preceding section, one has that

$$
\begin{aligned}
Z_4 &\leq C\sum_{s_1=1}^{s_N}\sum_{s=1}^{s_N-s_1}\sum_{l=1}^{s}\sum_{g=0}^{l}\sum_{k',k_1} s_1\mathbf{N}(s_1-1,s_1-k_1)\mathbf{N}(s_N-s_1,k'-k_1,s-1) \\
&\quad \frac{1}{g!}\left(\frac{s_N^2}{N}\right)^g\binom{l}{g}\left(\frac{2\gamma_N^{-1}s_N}{N}\right)^g\binom{s}{l}\left(\frac{2\gamma_N^{-2}\hat{C}_4 s_N}{N}\right)^{s-l}\pi_1^s\gamma_N^{k'-s_N}\sigma^{2s_N} \quad (36)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sigma^{2s_N}\sum_{s_1=1}^{s_N}\sum_{s=1}^{s_N-s_1}\sum_{g,k',k_1} s_1\mathbf{N}(s_1-1,s_1-k_1)\mathbf{N}(s_N-s_1,k'-k_1,s-1) \\
&\quad \frac{1}{g!}\left(\frac{s_N^2}{N}\right)^g\exp\left(\frac{\hat{C}_4' ss_N}{N}\right)\pi_1^s\gamma_N^{k'-s_N} \quad (37)
\end{aligned}
$$

where we used the fact that $l \leq s$. C and $\hat{C}_4$ are positive constants independent of N but $\hat{C}_4$ depends on the fourth moments of the entries of X; $\hat{C}_4' > 0$ is

another constant depending on $\hat{C}_4$. In the case where $s_N = O(\sqrt{N})$, one can readily see that

$$
\begin{aligned}
(37) \quad &\leq \quad C\sigma^{2s_N} \exp(\frac{(\hat{C}_4' + 1)s_N^2}{N}) \sum_{s_1=1}^{s_N} \sum_{s=1}^{s_N - s_1} \sum_{k', k_1} s_1 \mathbf{N}(s_1 - 1, s_1 - k_1) \\
&\quad \mathbf{N}(s_N - s_1, k' - k_1, s - 1)\pi_1^s \gamma_N^{k' - s_N} \\
&= \quad \hat{C}_4' Z_1.
\end{aligned}
$$

In the scale $s_N \sim N^{2/3}$, the above estimate needs to be refined (since $s_N^2 >> N$). In fact, we can improve the bound on the number of choices of the starting instants of the subpaths Q_{i_2}. To see this, call $t_1' < t_2' < \cdots < t_g'$ the instants corresponding to the end of the Q_{i_2}'s. Denote by $t_1 < t_2 < \cdots < t_g$ the instants beginning the reading of the Q_{i_2}'s. By definition of the gluing process, each edge started at the instant t_i is an up edge. Furthermore, if x'' denotes the Dyck path associated to the path P'', one has for any $i = 1, \ldots, g$ that $x''(t) \geq x''(t_i) > 0, \forall t \in [t_i, t_i']$. Thus, the interval $[t_i, t_i']$ is included in one sub-Dyck path of x''. We claim that the total number of ways to choose t_i and t_i' does not exceed $Cs_N^{3/2}$ for some constant $C > 0$. Indeed, choosing t_i' determines the sub-Dyck path of x'' containing $[t_i, t_i']$. We call X_j this sub-Dyck path, $2L_j$ its length and k_j the number of its odd up steps. Our estimate is obvious in the case where $L_j \leq s_N^{1/2}$. If $L_j \geq s_N^{1/2}$, we call $N(t_i')$ the number of ways to determine t_i. Let $\mathbb{E}_{L_j, k_j}$ denote the expectation with respect to the uniform distribution on the set χ_{L_j, k_j} of Dyck paths of length $2L_j$ with k_j odd up steps. Then there exists some constant $C > 0$ independent of N, k_j and L_j such that (for typical k_j's)

$$
\mathbb{E}_{L_j, k_j} \left(N(t_i')/s_N^{1/2} \right) \leq C. \tag{38}
$$

The above bound essentially follows from the estimation obtained in Section 2.5 in [20]. Indeed, setting $T_{0,n,k} := \#\chi_{n,k}$ for any n, k, one has that

$$
\mathbb{E}_{L_j, k_j} \left(N(t_i') \right) \leq \sum_{n, k'} \frac{4 \inf\{n, (L_j - n)\} T_{0,n,k'} T_{0,L_j-n,k_j-k'}}{T_{0,L_j, k_j}} \leq Cs_N^{1/2}.
$$

The term $4 \inf\{n, (L_j - n)\}$ counts the number of ways to choose t_i once given t_i' and $2n$ which is the length of the sub-Dyck paths between t_i and the first return to $x(t_i)$ followed by a down step. Given $g > 0$ and a Dyck path X of length $2L$, we set

$$
K_N^{\otimes g}(X) := \sum_{1 \leq t_1' < t_2' < \cdots < t_g' \leq 2L} \prod_{i=1}^{g} N(t_i'),
$$

where the sum bears on t_i' such that $X(t_i') > 0, \forall i = 1, \ldots, g$. Similarly and using the Appendix in [22], one can show that there exists a constant $C > 0$ independent of N, k_j and L_j such that (for typical k_j's)

$$
\mathbb{E}_{k_j, L_j} \left[K_N^{\otimes g}(X_j) \right] \leq \left(Cs_N^{3/2} \right)^g. \tag{39}
$$

31

In the sequel, we call $\chi_{s_N-s_1,k'-k_1,s-1}$ the set of Dyck paths $\tilde{x}$ in $\chi_{s_N-s_1,k'-k_1}$ with $s-1$ returns to 0. Let $\mathbb{E}_{k',s,s_N}$ denote the expectation with respect to the uniform distribution on Dyck paths of length $2s_N$ with s returns to the level 0 and k' odd marked instants. Let x_1 be the Dyck path defining the first return to 0 of x''. The estimate (36) may then be replaced by

$$\sigma^{2s_N}\sum_{s_1=1}^{s_N}\sum_{s=1}^{s_N-s_1}\sum_{l=1}^{s}\sum_{g=0}^{l}\sum_{k',k_1}\sum_{x_1\in\chi_{s_1-1,s_1-k_1}}s_1\,\frac{\gamma_N^{k'}}{\gamma_N^{s_N}}\sum_{\tilde{x}\in\chi_{s_N-s_1,k'-k_1,s-1}}\frac{C}{g!}$$

$$\mathbb{E}_{k',s,s_N}\left[\left(\frac{K_N^{\otimes g}(x_1\cup\tilde{x})}{N}\right)^g\right]\binom{l}{g}\left(\frac{2s_N}{\gamma_N N}\right)^g\binom{s}{l}\left(\frac{2\hat{C}_4 s_N}{\gamma_N^2 N}\right)^{s-l}\pi_1^s. \qquad (40)$$

We then deduce that there is a constant $\hat{C}_4' > 0$ (depending on $\hat{C}_4$) such that $(40) \leq \hat{C}_4' Z_1$. This ends the proof of Proposition 3.4. $\square$

Remark 3.1. Note that we have also shown that paths P' having at least one return to the vertex 1 on the bottom line at some positive level lead to a negligible contribution in any scale $s_N << N^{2/3}$. This follows from (39).

To complete the analysis, there remains to consider paths P leading to a glued path P' having some 1-edges which occur at positive levels and for which some of the erased edges e in-between P' and P'' appear 4 times or more in P (or in P'). We call Z_5 their contribution to the expectation (12).

Proposition 3.5. *One has that $Z_5 = o(1)Z_1$.*

Proof of Proposition 3.5: Consider the set of paths P such that the edges which are erased in-between P' and P'' and which arise at least four times in P do not appear in P''. We call Z_5^0 their contribution to Z_5 and set $\hat{Z}_5 := Z_5 - Z_5^0$.

We first show that $Z_5^0 << Z_1$. Let $e = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$ be one of the erased edges in-between P' and P'' which appears 4 times or more in P' (or in P). Denote by $n_e > 1$ the number of times where e is an erased edge in the gluing process. We here assume that P contributes to Z_5^0 so that e appears exactly $2n_e$ times in P. Thus there are n_e pairs of paths (Q_{i_1}, Q_{i_2}) in P' such that the derived glued subpath $Q_{i_1}\vee Q_{i_2}$ is associated to the same edge e.

Let us first prove that this decreases the number of preimages of the path P'' by a factor of

$$\left(\frac{4C(\nu_N+T_N)}{s_N}\right)^{n_e-1} \qquad (41)$$

where the quantities $\nu_N = \nu_N(P'')$ and $T_N = T_N(P'')$ have already been defined and used in the proof of Proposition 3.3. For this, consider the first (resp. second) of the previous considered pairs (Q_{i_1}, Q_{i_2}) and denote by $t_{e,1}$ (resp. $t_{e,2}$) the instant where Q_{i_2} begins and by $t'_{e,1}$ (resp. $t'_{e,2}$) the instant where Q_{i_2} ends in $Q_{i_1}\vee Q_{i_2}$. Suppose now that the instants $t_{e,1}$ and $t'_{e,1}$ have been chosen

and that the vertex α (resp. β) occurs at time $t_{e,1}$ (resp. $t'_{e,1}$) (the other case can be handled similarly). We claim that the number of choices for the instant $t'_{e,2}$ is at most of $2(\nu_N + T_N)$ instead of s_N. We recall that the quantities ν_N and T_N are such that, given an arbitrary vertex $v \neq 1$ in P'', there are at most $\nu_N + T_N$ up steps having v for endpoint. The announced bound readily follows since the edge started at $t'_{e,2}$ has α or β (already determined by the choices of $t_{e,1}$ and $t'_{e,1}$) as left endpoint. Obviously, the reasoning also applies to the $n_e - 2$ remaining glued subpaths $Q_{i_1} \vee Q_{i_2}$ having e as associated erased edge.

We now consider the weight of the path P with respect to that of P''. The weight of the erased edge e must multiply that of P'' : we use the fact that

$$\mathbb{E}|X_e|^{2n_e} \leq \sqrt{\mathbb{E}|X_e|^4}\sqrt{\mathbb{E}|X_e|^{4n-4}} \leq (Cn_e)^{n_e-1}.$$

Combining all the preceding, we deduce that Z_5^0 is of order

$$(40) \times \sum_{n\geq 1} \frac{1}{n!}\left(\frac{4C(\nu_N + T_N)n}{s_N}\right)^n \leq \frac{C(\nu_N + T_N)}{N^{2/3}} \times (40).$$

Z_5^0 is thus negligible with respect to Z_4 (and so Z_1) since $\nu_N + T_N << \sqrt{s_N}$ in typical paths P'' (cf. [20], Section 3.2).

We now estimate $\hat{Z}_5$. Consider a path P which contributes to $\hat{Z}_5$ and let e be an erased edge in-between P' and P'' which appears 4 times or more in P'. We also denote by $n_e \geq 1$ the number of times where e is an erased edge in the gluing process. As in typical paths P'' each edge is passed twice, e appears exactly $2(n_e+1)$ times in P. The previous reasoning works allowing $n_e \geq 1$. We then see that the sole case which remains to be considered is $n_e = 1$ that is when e occurs 4 times and not more in P. In this case, we determine the instants $t_{e,1}$ and $t'_{e,1}$ as in the estimate of Z_4. This determines the edge $e = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Now, the knowledge of e decreases the possible choices of the marked occurrence of e in P''. More precisely, one pays a cost of order $(\nu_N + T_N)^2/s_N$ so that a marked occurrence of β (for instance) arises in P'' after an occurrence of α (see also [24], p. 13). Using the above it is not hard to deduce that $\hat{Z}_5 << Z_1$. This finishes the proof of Proposition 3.5. $\square$

4 Higher moments

Let K be a fixed integer. Let also $c_i, i = 1, \ldots K$, be some positive real numbers. In this section, we compute moments of the type $\mathbb{E}\left(\prod_{i=1}^K \mathrm{Tr} V_N^{s_N^{(i)}}\right)$, where $(s_N^{(i)})$ are some sequences of integers such that $\lim_{N\to\infty} s_N^{(i)}/N^{1/2} = c_i$ if $\pi_1 > w_c$ and $\lim_{N\to\infty} s_N^{(i)}/N^{2/3} = c_i$ if $\pi_1 \leq w_c$. Then we prove the following result. Set

$$\tilde{V}_N^{(G)} = \frac{V_N^{(G)}}{u_+} \text{ if } \pi_1 \leq w_c \text{ and } \tilde{V}_N^{(G)} = \frac{V_N^{(G)}}{\tau(\pi_1)} \text{ if } \pi_1 > w_c.$$

Proposition 4.1. *Under the assumptions of Theorems 1.5 and 1.6, there exists a constant $C = C(K) > 0$ such that $\mathbb{E}\left(\prod_{i=1}^{K} \mathrm{Tr}\, \tilde{V}_N^{s_N^{(i)}}\right) \leq C$ and*

$$\mathbb{E}\left(\prod_{i=1}^{K} \mathrm{Tr}\, \tilde{V}_N^{s_N^{(i)}}\right) = (1 + o(1))\mathbb{E}\left(\prod_{i=1}^{K} \mathrm{Tr}\left((\tilde{V}_N^G)^{s_N^{(i)}}\right)\right).$$

In the case where $\pi_1 \geq w_c$, the constant C also depends on $\max_j \mathbb{E}|X_{1j}|^4$.

Proof of Proposition 4.1: We consider the variance $\mathbb{E}\left(\mathrm{Tr}V_N^{s_N} - \mathbb{E}\mathrm{Tr}V_N^{s_N}\right)^2$ only. Indeed the computations needed to consider higher moments follow from the same arguments combined with those developed in Section 5 of [25]. Proposition 4.1 can then be restated as follows.

Lemma 4.1. *There exists $C > 0$ such that $\mathbb{E}\left(\mathrm{Tr}\tilde{V}_N^{s_N} - \mathbb{E}\mathrm{Tr}\tilde{V}_N^{s_N}\right)^2 \leq C$ and one has $\mathbb{E}\left(\mathrm{Tr}\tilde{V}_N^{s_N} - \mathbb{E}\mathrm{Tr}\tilde{V}_N^{s_N}\right)^2 = (1 + o(1))\mathbb{E}\left(\mathrm{Tr}(\tilde{V}_N^G)^{s_N} - \mathbb{E}\mathrm{Tr}(\tilde{V}_N^G)^{s_N}\right)^2.$*

Proof of Lemma 4.1: Let us define $Y := \Sigma^{1/2}X$. Then,

$$p^{2s_N}\mathbb{E}\left(\mathrm{Tr}V_N^{s_N} - \mathbb{E}\mathrm{Tr}V_N^{s_N}\right)^2$$

$$= \sum_{P_{(1)}, P_{(2)}}^{*} \mathbb{E}\left(\prod_{(i,j)\in P_{(1)}} \hat{Y}_{ij} \prod_{(i,j)\in P_{(2)}} \hat{Y}_{ij}\right) - \mathbb{E}\left(\prod_{(i,j)\in P_{(1)}} \hat{Y}_{ij}\right)\mathbb{E}\left(\prod_{(i,j)\in P_{(2)}} \hat{Y}_{ij}\right).$$

Here, given an edge $e = (i,j) \in P_{(1)}$ (this is similar for $P_{(2)}$), $\hat{Y}_{ij}$ stands for Y_{ij} if e occurs at an odd instant of $P_{(1)}$ and for $\overline{Y_{ji}}$ if it occurs at an even instant. The starred sum bears on paths $P_{(1)}, P_{(2)}$ of length $2s_N$ sharing at least one common edge (i,j), $i \in [1, \ldots, N], j \in [1, \ldots, p]$. This follows from the fact that the Y_{ij}'s are independent centered random variables. We say that such paths are *correlated paths*. The contribution to the variance from correlated paths without 1-edges can be deduced from [20]. We thus focus on the pairs of correlated paths with 1-edges and assume without loss of generality that $P_{(1)}$ has at least one 1-edge.

We first consider the case where both $P_{(1)}$ and $P_{(2)}$ have 1-edges. We denote by T_1 (resp. T_2) the number of pairs of 1-edges in $P_{(1)}$ (resp. $P_{(2)}$). We also set $s = T_1 + T_2$. We build from $P_{(1)}$ (resp. $P_{(2)}$) T_1 (resp. T_2) subpaths $(P_i)_{1\leq i\leq T_1}$ (resp. $(P_i)_{T_1+1\leq i\leq s}$) starting and ending with a 1-edge as in Section 2.3. In the following, we use the denomination "1-subpath" or simply "subpath" of $P_{(1)}$ or $P_{(2)}$ to refer to some subpath P_i. By definition, the origin of $P_{(1)}$ (resp. $P_{(2)}$) occurs at some even instant in the subpath P_1 (resp. P_{T_1+1}). We concatenate the subpaths $(P_i)_{1\leq i\leq s}$ in the order they appear which leads to an even path P of length $4s_N$.

Case 1: The subpaths P_i in $P_{(1)}$ and those of $P_{(2)}$ share 1-edges only. Then, as in Section 2.3, we define the $l \leq s$ clusters of P and we apply the first gluing procedure yielding a path P' of length $2(2s_N - (s-l))$. We denote by x' the trajectory of P' and by m the number of returns to 0 of x'. As $P_{(1)}$ and $P_{(2)}$ are correlated, one has that $l < s$. Here, we will also assume that P' is fundamental that is $m = l$. Otherwise, this implies to perform the second gluing procedure on P' yielding a new path P'' but as we assume that the paths $P_{(1)}$ and $P_{(2)}$ share 1-edges only, all the arguments we will give to determine $P_{(2)}$ from P' are exactly the same when dealing with P'' (see below). Thus, focusing on P', $s-m$ counts the number of pairs of 1-edges that have been erased through the first gluing process. For the sequel, it is convenient to denote by $2L_j$, $j = 1, \ldots, m$ the length of the successive m sub-Dyck paths of x' ($\sum_j 2L_j = 2(2s_N - (s-l))$). To reconstruct $P_{(1)}$ and $P_{(2)}$ from P', one has to determine the $s - m$ instants of time where a 1-edge has been erased and reorder the subpaths thus defined. One also has to determine the origins of $P_{(1)}$ and $P_{(2)}$. By construction, the origin of $P_{(1)}$ occurs at some even instant in the first 1-subpath in P'. We call t_e the first moment of time where a 1-subpath of $P_{(2)}$ is glued to a 1-subpath of $P_{(1)}$. We call Q the latter subpath of $P_{(2)}$. One can note that at time t_e, a 1-edge which we call e is erased. Last, we let t_f be the instant where Q stops in P'. Two cases must be considered now since t_f can be an instant where a 1-edge is erased or where x' returns to 0.

Assume first that t_f is an instant where a 1-edge is erased. Assume that t_f and all but t_e of the $s - m - 2$ other moments of time where a 1-edge is erased have been selected in P'. Assume also that the corresponding $(s-1)$ 1-subpaths have been reordered. There are $\binom{s_N}{s-m-1}$ possible choices for the $s - m - 1$ instants of the erased edges and $(s - 1)!/m!$ ways to reorder the 1-subpaths thus defined. Indeed, the m subpaths beginning the sub-Dyck paths of x' arise in the same relative order in P' and in the concatenation P (cf. the proof of Lemma 2.1). We call $\hat{P}$ the path obtained after rearranging these $s-1$ subpaths. We now choose along $\hat{P}$ the instant t_0 defining the origin of $P_{(2)}$: this determines all the subpaths of $P_{(2)}$ except Q. There are at most $2s_N$ choices for t_0. A crucial fact now is that the knowledge of t_0 combined with that of t_f determines the instant t_e (in $\hat{P}$) since as $P_{(2)}$ is of length $2s_N$, the length of Q is then known. To obtain the full path $P_{(2)}$ and the final concatenation P, it remains to insert Q in $\hat{P}$; there are at most $2s$ ways to do this (the factor 2 comes from the choice of the direction of reading Q in P). Set $\tilde{s}_N = 2s_N - 1$ and $\hat{C}_4 := 1 + \max_v \mathbb{E}|X_{1v}|^4/\sigma^2$. Combining the whole, we get (for details, see the computations of Z_2 and Z_3 made in Section 3.1) that the contribution $Z_{v,1}^{(1)}$ to the variance from correlated paths $(P_{(1)}, P_{(2)})$ such that t_f is an instant where a 1-edge is erased is at most (for some constant $C > 0$)

$$Z_{v,1}^{(1)} \leq C\sigma^{2\tilde{s}_N} \sum_{s_1=1}^{\tilde{s}_N} \sum_{s=1}^{\tilde{s}_N - s_1} \sum_{m=1}^{s-1} \sum_{k=1}^{\tilde{s}_N} \sum_{k_1 \leq k+s-m} s_1 \frac{\gamma_N^k}{\gamma_N^{\tilde{s}_N}} \mathbf{N}(s_1 - 1, s_1 - k_1)$$

35

$$\mathbf{N}(\tilde{s}_N - (s - m) - s_1, k - k_1, m - 1)\frac{\left(\frac{2\hat{C}_4 s s_N}{N}\right)^{s'}}{s'!}\pi_1^s \frac{4 s s_N \hat{C}_4}{N} \quad (42)$$

$$:= \frac{4 C \hat{C}_4 s_N^{3/2}}{N}\sigma^{2\tilde{s}_N} \times A_{\tilde{s}_N} \quad (43)$$

where we let $s' := s - m - 1$. The factor $\hat{C}_4/N$ comes from the weight of the erased 1-edge e (it can indeed be shown that e can only occur at most four times in typical paths P). In the case where $s_N = O(\sqrt{N})$ and $\pi_1 > w_c$, we readily deduce that $(42)/(\tau(\pi_1))^{2s_N}$ is bounded (universality is discussed at the end of this section). In the case where $\pi_1 \leq w_c$ and $s_N = O(N^{2/3})$, it is a small computation, using the same arguments as in Lemma 3.1, to check that

$$A_{\tilde{s}_N} = O(1)\frac{u_+^{\tilde{s}_N}}{\sqrt{\tilde{s}_N}} \text{ if } \pi_1 < w_c \text{ and } A_{\tilde{s}_N} = O(1)u_+^{\tilde{s}_N} \text{ if } \pi_1 = w_c. \quad (44)$$

Assume now that the instant t_f is such that $x'(t_f) = 0$. We then fix t_f by choosing one such instant : this fixes some $1 \leq j \leq m$. Assume that all but t_e of the $s - m - 1$ other moments of time where a 1-edge is erased have been selected in P'. The knowledge of t_f and of the $s - m - 1$ selected instants determines the subpath Q_0 in P' which still has to be split into a subpath of $P_{(1)}$ and the first subpath, which we call Q, of $P_{(2)}$ that is glued to one subpath of $P_{(1)}$. By construction, Q_0 is included in the sub-Dyck path of x' ending at time t_f so that the length of Q_0 is not greater than $2L_j$. As before we reorder the $(s - 1)$ 1-subpaths thus defined to get the path $\hat{P}$. Now given $\hat{P}$, we claim that there are at most $8s_N$ different ways to choose t_f and the instant t_0 defining (along $\hat{P}$) the origin of $P_{(2)}$. Indeed, in the final concatenation (that is in P), the origin of $P_{(2)}$ is encountered along the first subpath of $P_{(2)}$. So in $\hat{P}$, t_0 is either in Q or in a 1-subpath which begins in the interval of time $[2s_N, 2s_N + 2L_j]$. Denoting by $2l'$ the length of the 1-path beginning in $[2s_N, 2s_N + 2L_j]$ but which does not finish in this interval (if it exists), there exists some L'_j such that $2l' \leq 2L'_j$. Hence the number of possible choices for t_f and t_0 is at most $\sum_{j=1}^m 2L_j + 2L'_j \leq 8s_N$ which is what we wanted. We then readily conclude that the contribution $Z_{v,2}^{(1)}$ of such correlated paths $(P_{(1)}, P_{(2)})$ behaves as $Z_{v,1}^{(1)}$ since it is at most 4 times the r.h.s of (42).

Case 2: The paths $P_{(1)}$ and $P_{(2)}$ share edges which are not 1-edges. We denote by $Z_{v,1}^{(2)}$ the contribution of such correlated paths $(P_{(1)}, P_{(2)})$ to the variance. Dealing with such a pair, we still apply the first gluing procedure on the concatenation P. If the path P' obtained in this way is such that all the 1-edges arise when the trajectory of P' returns to the level 0, we can finish the proof as before.

Otherwise, we apply the second gluing procedure defined in Section 3.2 getting a new path P'' where each occurrence of the vertex 1 on the bottom line corresponds to a marked instant or an instant where its trajectory x'' returns to

0. We denote by m the number of returns to 0 of x'' and by $2L_1, \ldots, 2L_m$ the length of the successive m sub-Dyck paths of x''. Given P'', we shall now reconstruct the paths $P_{(1)}$ and $P_{(2)}$. As before, we call Q the first subpath of $P_{(2)}$ that is glued to one of $P_{(1)}$. We consider here the case where Q is glued using the second procedure which means that its gluing is associated to a marked occurrence of the vertex 1 in P'' (since the other case can be treated using the arguments developed in the previous case). We also denote by t_e (resp. t_f) the instant of time where Q begins (resp. ends) in P''. We assume that all the instants needed to define the gluing but that of Q are chosen. We also assume that all the marked occurrences of 1, except that associated to the gluing of Q, are known. All these instants define $(s-1)$ 1-subpaths which we reorder as before defining a new path called $\hat{P}$. Now if one also knows the instant t_0 defining the origin of $P_{(2)}$ in $\hat{P}$ and if t_e is fixed, then the length of Q is determined and there is no choice for t_f. In the sequel we set $l_Q = t_f - t_e$. We consider the case where the cluster containing the subpath Q is well separated from the others (not interlaced). The other case follows from the same considerations. Then during the time interval $[t_e, t_e + l_Q]$, the trajectory x'' of P'' does not go below the level $x''(t_e)$ and there is also a marked occurrence of 1 on the bottom line. Furthermore, by the definition of the second gluing procedure, $[t_e, t_e + l_Q]$ is included in a sub-Dyck path of P''. Assume that this is the jth sub-Dyck path, which is thus of length $2L_j$. Let also k_j be the number of odd up steps in this sub-Dyck path. Denote by N_{t_e} the total number of possible choices for the instants t_e and that of the marked occurrence of 1 associated to the gluing of Q. Let $\mathbb{E}_{L_j,k_j}$ denote the expectation with respect to the uniform distribution on the set χ_{L_j,k_j} of Dyck paths of length $2L_j$ with k_j odd up steps. Then there exists a constant $C > 0$ independent of N, k_j and L_j such that (for typical k_j's)

$$\mathbb{E}_{L_j,k_j}\left(N_{t_e}/s_N^{3/2}\right) \leq C. \tag{45}$$

The above bound essentially follows from arguments close to those used in (38) and the estimation obtained in Section 2.5 in [20]. More precisely, setting $T_{0,n,k} := \#\chi_{n,k}$ for any n, k, it is easy to show that

$$\mathbb{E}_{L_j,k_j}\left(N_{t_e}\right) \leq \sum_{n,k'} \frac{4n(L_j - n)T_{0,n,k'}T_{0,L_j-n,k_j-k'}}{T_{0,L_j,k_j}} \leq C s_N^{3/2}$$

where n (resp. $L_j - n$) counts the number of possible choices of the instant of the marked occurrence of 1 (resp. of the instant t_e) if the sub-Dyck path between t_e and the first return to $x(t_e)$ followed by a down step is of length $2n$. The above estimate clearly holds if $L_j \leq s_N^{1/2}$ and if $L_j \geq s_N^{1/2}$, one can copy the arguments of Section 2.5 in [20]. We are now in position to estimate the contribution $Z_{v,1}^{(2)}$. To this aim, we denote by Z_4' the contribution Z_4 of Section 3.2 corresponding to even paths of length $4s_N$ instead of $2s_N$. Apart from N_{t_e}, one needs to multiply the contribution of P'' by a factor of the order $16\sigma^2 s s_N/N^2$. Indeed, the number of ways to determine the sub-Dyck path of P'' where $[t_e, t_e + l_Q]$ is

included and the origin of $P_{(2)}$ in $\hat{P}''$ may be controlled as before by a factor $\sum_{j=1}^{m} 2L_j + 2L'_j \leq 8s_N$. Besides, there are at most $2s$ ways to insert Q in $\hat{P}$ and choose its orientation. Last, due to the edge e erased at time t_e in between P and P'' (it can be shown that e does not occur in typical paths P'' and occurs twice in typical paths P) and due to the marked occurrence of 1 associated to t_e and t_f, the weight of the path has to be multiplied by a factor $\sigma^2/p \times 1/N$. Hence, inserting the factor $\sigma^2 s s_N N_{t_e}/(pN)$ in the computations of Z'_4 and using (45) and (44), leads to

$$Z_{v,1}^{(2)} \leq \frac{C\sigma^2 s_N^{3/2}}{N} \times Z_{v,1}^{(1)},$$

for some positive constant C. One can then check that $Z_{v,1}^{(2)} = O(1)Z'_4$.

To complete the analysis, we now investigate the case where $P_{(2)}$ has no 1-edge. In this case, we first apply the first gluing procedure to $P_{(1)}$, which leads to a path called $P'_{(1)}$. Then we use the second gluing procedure to "insert" $P_{(2)}$: we consider the first edge along $P'_{(1)}$ which is also encountered along $P_{(2)}$ and use it to glue $P_{(2)}$ by the construction procedure used in Section 3.2. Last we use the second gluing procedure (if needed) to obtain a final path where all the 1-edges arise at level 0 of the associated trajectory or correspond to a marked occurrence of 1. The procedure we use in this case can be compared to that of Case 2, provided the path $P_{(2)}$ is "assimilated" to a 1-subpath. The analysis performed in Case 2 can be copied up to minor modifications to show that the contribution to the variance of such correlated paths is of the order of $Z_{v,1}^{(2)}$.

Combining all the preceding implies that the total contribution to the variance $\mathrm{Var}\big(\mathrm{Tr}\tilde{V}_N^{2s_N} \big)$ from correlated paths $(P_{(1)}, P_{(2)})$ is bounded. (44) also implies that in the case where $\pi_1 < w_c$, the contribution of paths with 1-edges is negligible in the large N-limit.
To conclude to universality of the variance, we can use the fact that only the pairs of correlated paths with 1-edges seen at most 4 times and other edges passed exactly twice contribute in a non negligible way to (42). Thus universality of the variance (and higher moments) can be deduced from universality of the expectation of traces of $V_N^{2s_N}$. This finishes the proof of Proposition 4.1. $\square$

5 More than one eigenvalue greater than 1

In this section we consider more general spiked sample covariance matrices (V_N) given by (3) with a spiked covariance matrix $\Sigma = \mathrm{diag}(\pi_1, \pi_2, \ldots, \pi_r, 1, \ldots, 1)$ where $r \geq 2$ is some fixed integer independent of p and N and $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_r > 1$ are given real numbers independent of p and N also.
We shall explain the main modifications to be made in the previous analysis in order to prove Theorems 1.5 and 1.6 in this more complex case. As in the case where $r = 1$ (recall Section 1.2 which includes the case where $r \geq 1$), one has to prove boundedness and universality of moments (of any fixed order) of traces of

high powers of V_N. We here restrict ourselves to the study of the expectation. Universality of moments of higher order of traces of V_N then follows from the same arguments as in the case where $r = 1$ (see Section 4).

As before, (s_N) denotes a sequence of integers that may grow to infinity. In order to examine the contribution from paths P to the expectation $\mathbb{E}(\mathrm{Tr}V_N^{s_N})$, one has to consider the number of times each of the vertices $1, \ldots, r$ occurs on the bottom line of P. To fix the idea of the analysis, we consider the case where $r = 2$. The general case then follows from a straightforward extension of the arguments used when $r = 2$.

Let then a path P of length $2s_N$ contributing to $\mathbb{E}(\mathrm{Tr}V_N^{s_N})$ be given. We assume that P has T_1 (resp. T_2) pairs of 1-edges (resp. 2-edges) with $T_1+T_2 \geq 1$. We set $s := T_1 + T_2$. To deal with such a path, we define a glued path P'. The gluing procedure (leading to P') defined in Section 2.3 when P has only 1-edges (or only 2-edges) is modified in the following way when $T_1T_2 > 0$. We first identify the instants $t_1 < t_2 < \cdots < t_s$ where the first edge of pairs of 1-edges or 2-edges occur in the path. We call $e_i^{(l)}$ (resp. $e_i^{(r)}$) the left (resp. right) edge of these s pairs of edges. Then, for $i \geq 2$, we define the subpath P_i as the subpath starting with $e_{i-1}^{(l)}$ and ending at $e_i^{(r)}$. As before P_1 is the path starting at $e_s^{(r)}$ and ending at $e_1^{(l)}$ (we concatenate the end and beginning of P). Two subpaths P_i and $P_{i'}$ are now said to belong to the same "connected component" if they share a 1-edge or a 2-edge. We denote by l ($l \leq s$) the number of such connected components. Consider the first connected component and denote by l_1 its cardinality. We claim (since each of the 1- or 2-edges occurs an even number of times, see Subsection 2.3.2) that there exists a way to glue the l_1 subpaths in order to form a path satisfying the following conditions: it starts and ends with the same 1-edge or 2-edge and has no other 1-edge or 2-edge. We do the same for the other components in such a way to define l paths (corresponding to each connected component) which have pairwise distinct first edges and appear in the same relative order as in the initial path P. We denote by $Q_j, j = 1, \ldots, l$ the successive paths derived from the gluing process (in case $r = 1$, we denoted them by P_j^g). To each path Q_j, we associate its connected component (also called cluster) $\mathcal{S}_j, j = 1, \ldots, l$, which is the set of initial subpaths that have been glued to form Q_j. We then obtain a "path" P' of length $2(s_N - (s-l))$ with origin 1 or 2 and having m returns to the level 0, for some $m \leq s$. Note that (as $T_1T_2 > 0$) P' is not a path in the usual sense, since one might switch from vertex 1 to 2 at any instant where one switches from one cluster to another. Nevertheless each cluster (or subpath Q_j) starts and ends with the same vertex. We start with the following important remark. Assume that the clusters $\mathcal{S}_j$ and $\mathcal{S}_{j+1}$ do not have the same origin and that, for instance, $\mathcal{S}_{j+1}$ starts with a 2 (the reverse case is similar). This necessarily implies that some subpaths starting or ending with a 2 have been glued in some preceding clusters. In other words, if we denote by K the number of times one switches the origin of successive clusters, one has that $K \leq s - l$.

We first assume that $l = m$ that is the returns of the trajectory associated to P' to the level 0 define the $l = m$ clusters. Here we show that the contribution of paths for which $T_2 > 0$ is negligible if $\pi_2 < \pi_1$. Their contribution is of the same order as that of paths with only 1-edges (and only 2-edges) in the case where $\pi_1 = \pi_2$.

One of the main points in the analysis is to estimate the number of preimages P of a glued path P', that is to establish the counterpart of Lemma 2.1. The number of ways to determine the set $\mathcal{T}$ of the $s-m$ moments of time where some 1- or 2-edge has been erased is at most $\binom{s_N}{s-m}$ as before. Yet the number of ways to reorder the subpaths P_i thus defined is much smaller than in the case where $r = 1$. When $r = 1$, we used (recall the proof of Lemma 2.1) the upper bound $\frac{s!}{m!} \leq s^{s-m}$. When $r = 2$, there are some constraints on the way to reorder the subpaths P_i: they must be reordered in such a way to form a path. Indeed a subpath starting with a 1-edge (resp. 2-edge) cannot follow a subpath ending with a 2-edge (resp. 1-edge). When $r = 2$ (and $T_1 T_2 > 0$) and assuming that the origin of P' is chosen, the maximal number of ways to reorder the subpaths P_i if one does not take these constraints into account is bounded by s^{s-m}: for each cluster $\mathcal{S}_j$, it is enough to indicate the number of "slots" between the first subpath and each of the subpaths of $\mathcal{S}_j$. Let us call R the number of ways to reorder the P_i's in an admissible way now. Then if $T_1 T_2 > 0$, one has that

$$R \leq 8 s^{s-m-1}. \tag{46}$$

To prove this, we need a few notations. We call x_1 (resp. x_2) the number of subpaths P_i starting and ending with a 1-edge (resp. with a 2-edge). And $2x_3 := s - x_1 - x_2$ denotes the number of paths with both a 1-edge and a 2-edge. It will be convenient to call these paths respectively 1-paths, 2-paths or 12-paths. We here consider the set of paths for which $T_1 \geq T_2$ and which are obtained from an admissible configuration of the P_i's. Assume that $x_2 \neq 0$. If $x_1 \geq x_3$, consider all the configurations obtained by permuting one of the x_1 1-subpaths with one of the x_2 2-subpaths. Then distinct admissible configurations lead to distinct non admissible configurations. Similarly, if $x_3 \geq x_1$, we consider all the configurations obtained by permuting one of the x_3 12-subpaths with one of the x_2 2-subpaths. If now $x_2 = 0$, we consider all the configurations obtained by permuting one of the x_3 12-subpaths with one of the x_1 1-subpaths. In all these cases, the number of permutations is at least $s/8$ which leads to (46).

From now on, we assume that there exists a real number $c > 0$ such that $\lim_N s_N/\sqrt{N} = c$ if $\pi_1 > w_c$ and $\lim_N s_N/N^{2/3} = c$ if $\pi_1 \leq w_c$.
In the following, we focus on the estimation of the contribution from paths P having 1-edges and 2-edges passed only twice. As in the case $r = 1$ (and calling on [20]) it is not hard to see that, amongst the associated glued paths P', the typical ones have edges passed at most twice.
Thus, the contribution of such paths P with T_1 pairs of 1-edges and T_2 pairs of

2-edges $(T_1 \geq T_2 > 0)$ can be bounded from above by

$$C\sigma^{2s_N} \sum_{s_1=1}^{s_N} \sum_{s=0}^{s_N-s_1} \sum_{m=1}^{s} \sum_{K=1}^{m} 1_{K \leq s-m} \binom{m}{K} \sum_{k,k_1}$$
$$s_1 \mathbf{N}(s_1 - 1, s_1 - k_1) \mathbf{N}(s_N - s_1 - (s-m), k - k_1, m - 1)$$
$$\left(\frac{2\gamma_N^{-2}}{N}\right)^{s-m} \binom{s_N - (s-m) - 1}{s-m} (Cs)^{s-m-1} \gamma_N^{k+(s-m)-s_N} \pi_1^{T_1} \pi_2^{T_2}, \quad (47)$$

where the extra factor $\binom{m}{K}$ comes from the fact that we have to distribute the m starting points of clusters into those starting with 1's and 2's (and C is a positive constant whose value may vary in the following). The contribution of paths for which $T_2 \geq T_1 > 0$ can be analyzed in a similar way. One simply interchanges the role of x_1 and x_2 in the previous reasoning.

As $\sum_{K=1}^{m} 1_{K \leq s-m} \binom{m}{K} \leq 2^m$, it is clear that the contribution of the paths P such that $m \leq 100(s - m)$ (100 is an arbitrarily large constant here) yields a contribution which is at most in the order of

$$C\sigma^{2s_N} \sum_{s_1=1}^{s_N} \sum_{s=0}^{s_N-s_1} \sum_{m=1}^{\frac{100}{101}s} \sum_{K=1}^{m} 1_{K \leq s-m} \binom{m}{K} \sum_{k,k_1}$$
$$s_1 \mathbf{N}(s_1 - 1, s_1 - k_1) \mathbf{N}(s_N - s_1 - (s-m), k - k_1, m - 1)$$
$$\left(\frac{2\gamma_N^{-2}}{N}\right)^{s-m} \binom{s_N - (s-m) - 1}{s-m} (Cs)^{s-m} \gamma_N^{k+(s-m)-s_N} \pi_1^s \frac{1}{s} \sum_{T_2 \leq s} \left(\frac{\pi_2}{\pi_1}\right)^{T_2}$$
$$= \begin{cases} O(\frac{s_N}{N}) Z_1 & \text{if } \pi_2 < \pi_1, \\ O(1) Z_1 & \text{if } \pi_2 = \pi_1. \end{cases} \quad (48)$$

where Z_1 is given by (24).

There now remains to estimate the contribution of paths P for which $m > 100(s - m)$. To this aim, we need to refine our preceding reasoning. Assume that the $s - m$ moments of time of the set $\mathcal{T}$ as well as the K instants where one switches the origin of the clusters have been selected. Assume also for ease that the origin of the path P' is 1 and denote by m_1 (resp. m_2) the number of 1-paths (resp. 2-paths) starting a cluster. Last set $m_3 = m - m_1 - m_2$. To reorder the P_i's, we first reorder the $2x_3$ 12-paths. There are $\frac{(2x_3)!}{m_3!}$ ways to do so. Then we determine the number of 1-paths and 2-paths to be inserted in-between the 12-paths and reorder them. There are at most $\frac{x_1!}{m_1!} \frac{x_2!}{m_2!} 2^{s-m}$ ways to do so (the 2^{s-m} is due to the possible choice of the direction of reading each of the 1- and 2-paths).

Thus the contribution of paths P for which $m \geq 100(s - m)$ can be bounded from above by

$$C\sigma^{2s_N} \sum_{s_1=1}^{s_N} \sum_{s=0}^{s_N-s_1} \sum_{m \geq \frac{100}{101}s} \sum_{K=1}^{m} 1_{K \leq s-m} \binom{m}{K} \sum_{k,k_1}$$
$$s_1 \mathbf{N}(s_1 - 1, s_1 - k_1) \mathbf{N}(s_N - s_1 - (s-m), k - k_1, m - 1) \left(\frac{2\gamma_N^{-1}}{N}\right)^{s-m}$$

$$\binom{s_N - (s-m) - 1}{s - m} \frac{1}{m_1! m_2! m_3!} (Cs)^{s-m} \gamma_N^{k-s_N} \pi_1^s \sum_{T_2 \leq s} \left(\frac{\pi_2}{\pi_1} \right)^{T_2}. \quad (49)$$

Using the fact that $\frac{1}{m_1! m_2! m_3!} \leq \frac{1}{(m/3)!} \frac{3 \times 101/100}{s}$ and that $\sum_K \binom{m}{K} \leq 8^{m/3}$, it is clear that the contribution of paths for which $s \geq \sqrt{s_N}$ is negligible. The contribution of paths for which $s \leq \sqrt{s_N}$ can be analyzed as follows. If $\pi_1 > w_c$, their contribution is of order $u_+^{s_N} \pi_1^{\sqrt{s_N}} << \tau(\pi_1)^{s_N}$ and is thus negligible. If $\pi_1 \leq w_c$ it is not hard to see that their contribution is at most of order of Z_1 (and thus negligible if $\pi_1 < w_c$).

The contribution from paths P (such that $l = m$) having 1-edges and 2-edges possibly read more than twice can be examined by refining the above analysis thanks to arguments already used in Section 3 for the investigations of Z_3. We skip the detail. Thus, one can show that paths P such that $l = m$ satisfy:
(a) if $\pi_1 = \pi_2 \geq w_c$, the typical paths P have 1-edges and 2-edges seen at most 4 times and no other edge seen more than twice;
(b) if $\pi_1 > \pi_2$, the typical paths have no 2-edge;
(c) if $\pi_1 < w_c$, the typical paths have neither 1-edges nor 2-edges.
Last the contribution from paths P such that $l > m$ that is when some occurrences of 1- or 2-edges in P' arise at some positive level can be analyzed using the same arguments as in Section 3.2 and arguments as above. We then deduce that when $r = 2$, the typical paths contributing to $\mathbb{E}(\mathrm{Tr} V_N^{s_N})$ satisfy the three preceding conditions (a) to (c). Combining all the preceding justifies the universality of the expectation $\mathbb{E}(\mathrm{Tr} V_N^{s_N})$.

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